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Does the Global Leverage Dynamic Gravitate to an Invariant?

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Literature

- **Leverage and Financial Instability**
- *Voprosy Economiki*, #9, 1-27, 2012 (in Russian)
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- **Logistic Model of Financial Leverage**
- *HSE Economic Journal,vol.17, #4,*585-616, 2013 (in Russian)
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- **Financial Assets Collateralization and Stochastic Leverage**
- *HSE Economic Journal,vol.18, #2,* 183-215, 2014 (in Russian)
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- **Stochastic Leverage of the Global Financial System,**
- *Proceedings of XVI International April Conference*, NRU HSE, 732-741, Moscow, 2016
-
- **Stochastic Logistic Model of the Global Financial Leverage**,
- *The BE Journal of Theoretical Economics*, 2018, issue 1

The Global Debt Perspectives

Global debt in the middle of 2017, according to the *Institute of International Finance*, has reached $237 trillion - its highest level on record. Global financial assets became almost 4times larger than the world GDP ($76 trillion).

The current financial system is, evidently, bloated and even worse, it is unsustainable. Global debt would require around $20 trillion in reimbursements. Actual servicing of around $1.5 trillion would doom the global debt to blow up and ultimately burst.

But is this outcome unavoidable?

Debt as a Generic Relation

In a historical perspective, current problems are in no way new: debt accumulation has been going on for, hopefully the first, five thousand years through reimbursements, innumerable market corrections, defaults, forgiveness and crises.

As one of the generic human relations, debt *per se*, and in its complex conjunction with wealth, has been thoroughly investigated since the times of Plato and Aristotle; it attracted the renewed attention in the aftermath of the 2007-2008 "credit crunch".

Debt is known as the only "good" bought back by its seller (issuer) from the buyer (creditor) according to some specified contract.

The debt is, obviously, a global centripetal force while its reimbursement invokes mighty centrifugal tendencies. Thus, it is reasonable to expect that the space-time coherency, and other tenets of the Keynesian "practical theory of the future", would be embraced in *the modeling of a super-large (global) financial system in a very long run*.

Aggregate debt interacts in a complicated way with wealth in the long run: opposing each other the debt, after its reimbursement, became a part of additional wealth. As perpetuity, global debt is, obviously, an unaccomplished wealth. Its accumulation "is to produce results, or potential results, at a comparatively distant, and sometimes at an *indefinitely* distant, date" (Keynes, 1937, p.213).

The global debt and capital form a *huge closed system;* evolving in accordance with its own laws it dictates booms and busts throughout the world. Dwarfing the largest banks or non-financial corporations, global financial aggregates are devoid of idiosyncrasies like complicated international financial markets, cross-border banking flows, regional saving-dissaving asymmetries, etc.

Ages ago it was well understood that the debt behaviour was facilitated by the duality of interest rates and collateral ratios. Yet the modeling of interrelations between collateral ratios and rates of return was started fairly recently (Geanokoplos, 1999), while the logistic model of leverage dynamics (Smirnov, 2018) conveying interrelations between borrowers and creditors is to be studied properly.

The intertwined dynamics of **fungible capital and debt assets** were modeled via the duality of collateral ratios and rates of return. Differential equations (linear – for the capital ratio, and logistic - for the leverage) were transformed into stochastic diffusions for which the appropriate stationary forward Kolmogorov (Fokker-Plank) equations were solved with regard to their probability density functions.

The complicated long-term behaviour of aggregate borrowers and creditors was explained by the stationary gamma distribution with or without mode. The leverage gravitation towards its anchor followed the gradual evaporation of excessive debt; the implied variance and the stationary mode in a compression of fungible liabilities were formed by structural market reforms and needed no particular monetary policy.

Economic Invariants

"The fundamental goal of science is to find invariants", (Simon, 1990, p.1). Quantitative or qualitative invariants play an important role in explaining dynamics of natural, biological and psychological processes. In economics the systematic study of invariants might be traced back to the concept of the natural interest rate elaborated by the great Swedish economist K. Wicksell (1898).

Table 1. Global assets, debt, capital and some structural parameters

Years (1)	Assets, \$ tn (2)	Debt, \$ tn (3)	Capital, \$ tn (4)	GDP, \$tn (5)	Capital ratio, w_t (6)	Leverage, l_t (7)	Spread a_t (8)	Spread c_t (9)	Parameter, b_t (10)
1999	117.1	83.6	33.5	32.5	0.285	3.51			
2000	116.0	85.0	31.0	33.6	0.267	3.75	-0.025	-0.087	-0.007
2001	112.5	85.6	26.9	33.4	0.238	4.2	-0.039	-0.144	-0.011
2002	118.4	95.6	22.8	34.6	0.192	5.21	-0.064	-0.268	-0.015
2003	142.9	111.7	31.2	38.9	0.218	4.59	0.038	0.201	0.007
2004	162.5	125.8	36.7	43.8	0.226	4.43	0.011	0.051	0.002
2005	168.3	127.9	40.4	47.4	0.24	4.17	0.019	0.085	0.004
2006	194.4	144.4	50.0	51.4	0.258	3.88	0.027	0.113	0.007
2007	227.1	166.8	60.3	57.9	0.265	3.77	0.012	0.046	0.003
2008	204.0	171.7	32.3	63.5	0.158	6.33	-0.13	-0.494	-0.034
2009	229.4	184.8	44.6	60.2	0.194	5.15	0.048	0.305	0.008
2010	243.9	192.4	51.5	66.0	0.211	4.74	0.023	0.117	0.005
2011	244.6	200.2	44.4	73.3	0.182	5.5	-0.038	-0.18	-0.008
2012	263.0	211.9	51.1	74.9	0.194	5.16	0.017	0.091	0.003
2013	274.6	214.4	60.2	77.0	0.22	4.55	0.033	0.169	0.006
2014	275.9	212.4	63.5	79.1	0.231	4.33	0.014	0.064	0.003
2015	270.8	208.9	61.9	74.8	0.229	4.37	-0.003	-0.011	-0.001
2016	281.3	216.4	64.9	75.9	0.231	4.33	0.003	0.014	0.001
2017	316.4	237.2	79.2	80.7	0.251	3.98	0.029	0.124	0.007

{Sources: World Bank, Institute of International Finance, Author's estimations}

Annual empirical volatilities of the capital ratio and leverage were estimated as $\sigma_w = 0.007$ and $\sigma_l = 0.16$, respectfully.

Numerical simulations and graphs were performed using the Wolfram*Mathematica10* computing system.

Macrofinancial ratios

The aggregate financial system:

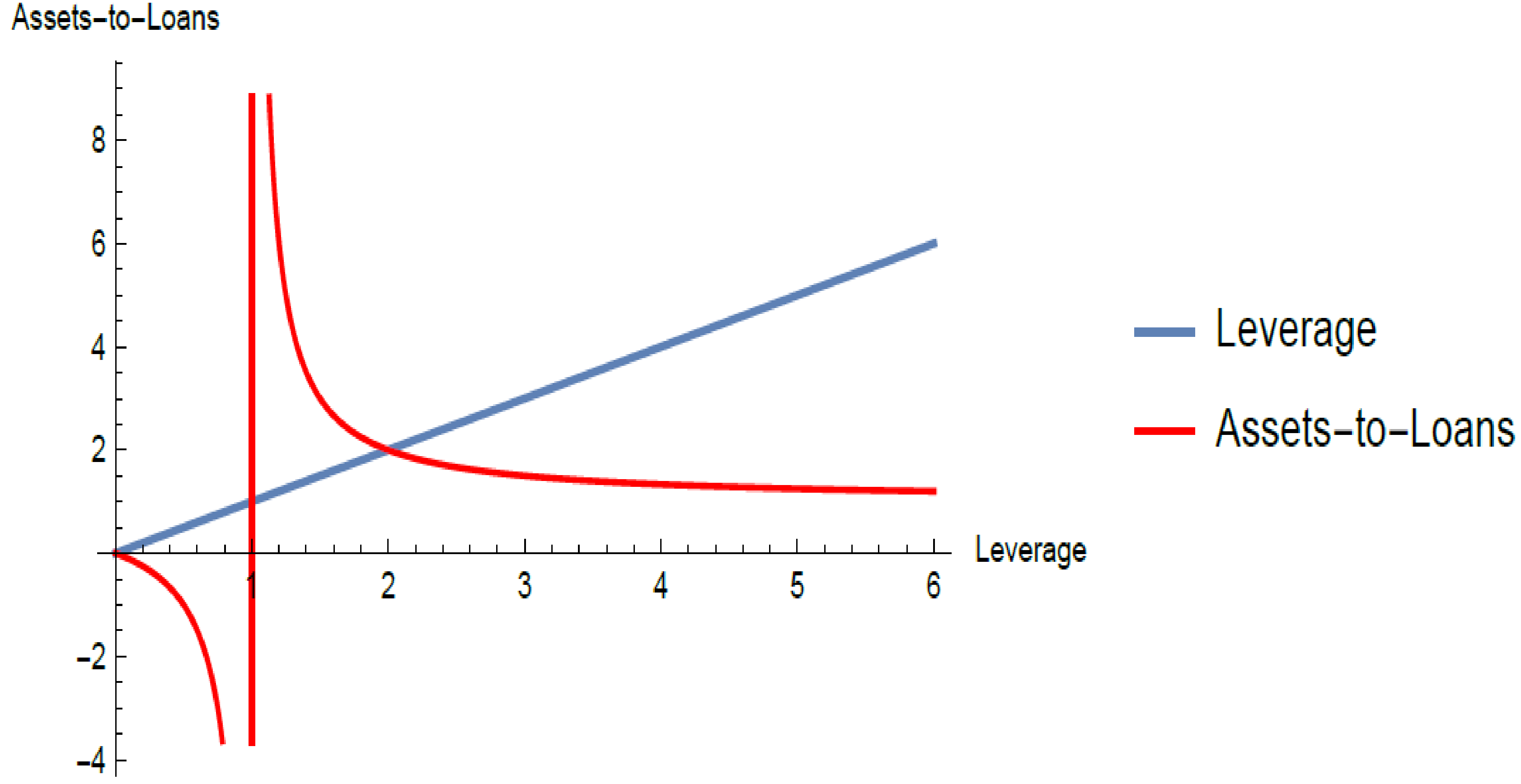
$$A(t) = x(t) + E(t)$$

where $A(t)$ is a value of total assets;
$x(t)$ is a value of total debt;
$E(t)$ is a value of total capital.

There are four basic ratios widely used in analysis and practice:
the collateral ratio, $cr(t) = A(t) / x(t)$;
the loan-to-value ratio, $z(t) = x(t) / A(t)$;
the margin (haircut) ratio, $w(t) = E(t) / A(t)$; and
the leverage, $l(t) = A(t) / E(t)$.

$$cr(t) * z(t) = 1; \quad z(t) + w(t) = 1; \quad w(t) * l(t) = 1.$$

Leverage and the Collateral Ratio



Financial flows and rates of return

The balance between aggregate financial flows:

$$d\,A(t) = d\,x(t) + d\,E(t)$$

where d is the operator of taking differentials.
Instantaneous rates of return:

on assets, $ROA \equiv \mu = dA(t)\,/\,A(t)dt$;

on aggregate debt, $ROR \equiv r = dx(t)\,/\,x(t)dt$;

on equity, $ROE \equiv \rho = dE(t)\,/\,E(t)dt$.

The balanced financial market condition

The balance of financial flows:

$$\mu A = r\,x + \rho\,E$$

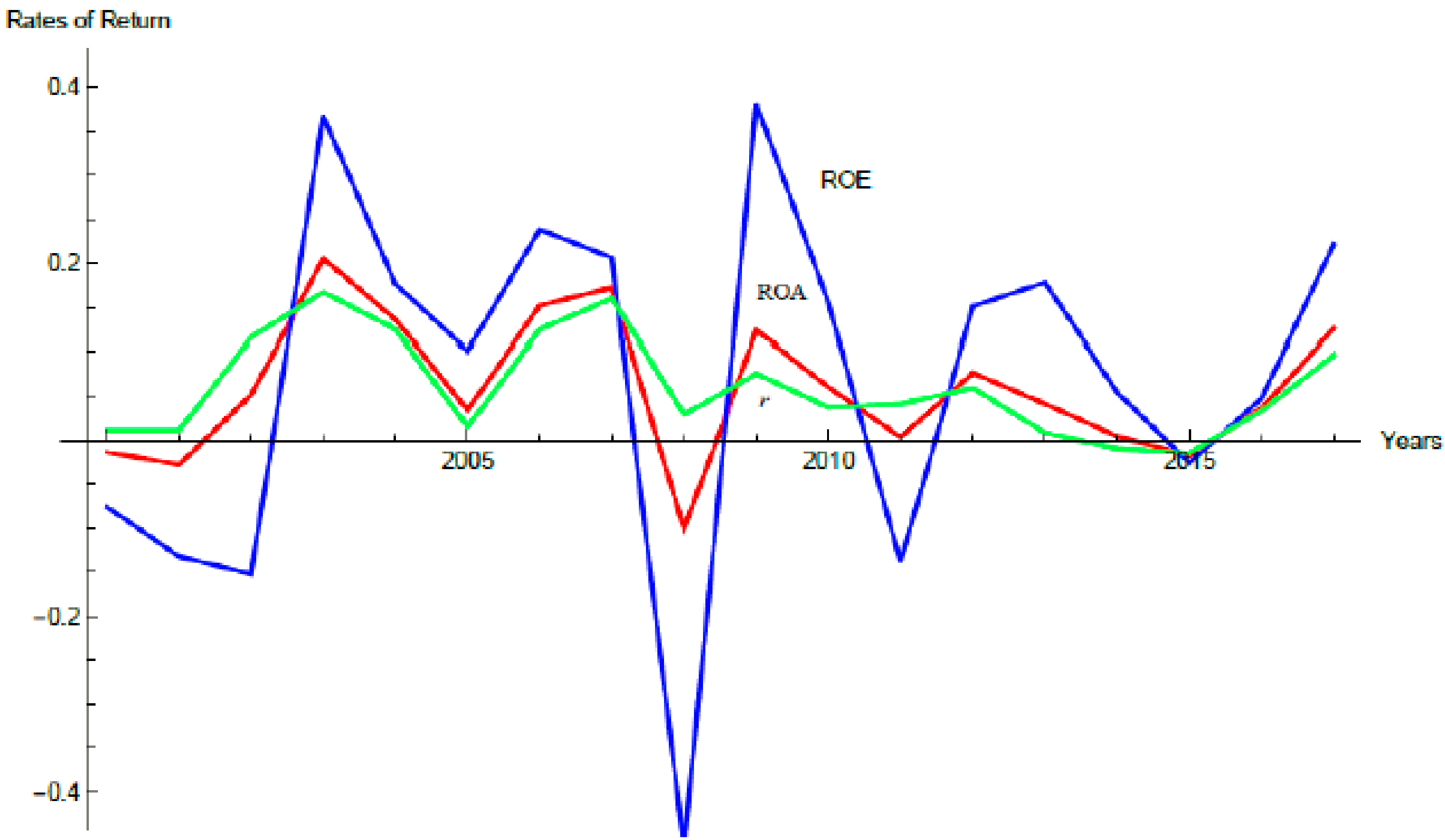
the macrofinancial LtV ratio, $z(t) = x(t)\,/\,A(t)$,
or the margin ratio, $w(t) = E(t)\,/\,A(t)$.

The balanced financial market (BFM) condition:

$$\mu = r z(t) + \rho(1 - z(t)).$$

$$\mu = r + (\rho - r)w$$

Figure I
Global rates of return in 1999-2017



Spreads and the short term debt demand and supply

The BFM condition is a static equality between indicators of (constant) demand for capital, $(\mu - r)$, and its supply, $(\rho - r)w$:

$$(\mu - r) = (\rho - r)w$$

.

The root is a relation of two spreads:

$$w^* = a / c \text{, or} \quad w^* = b / a$$

where $c = \rho - r$, and $b = a^2 / c$.

The ODE of the capital ratio dynamic

$$dw/dt = -aw(t) + a\ .$$

The economic reality is preserved for $(b/a) < 1$:

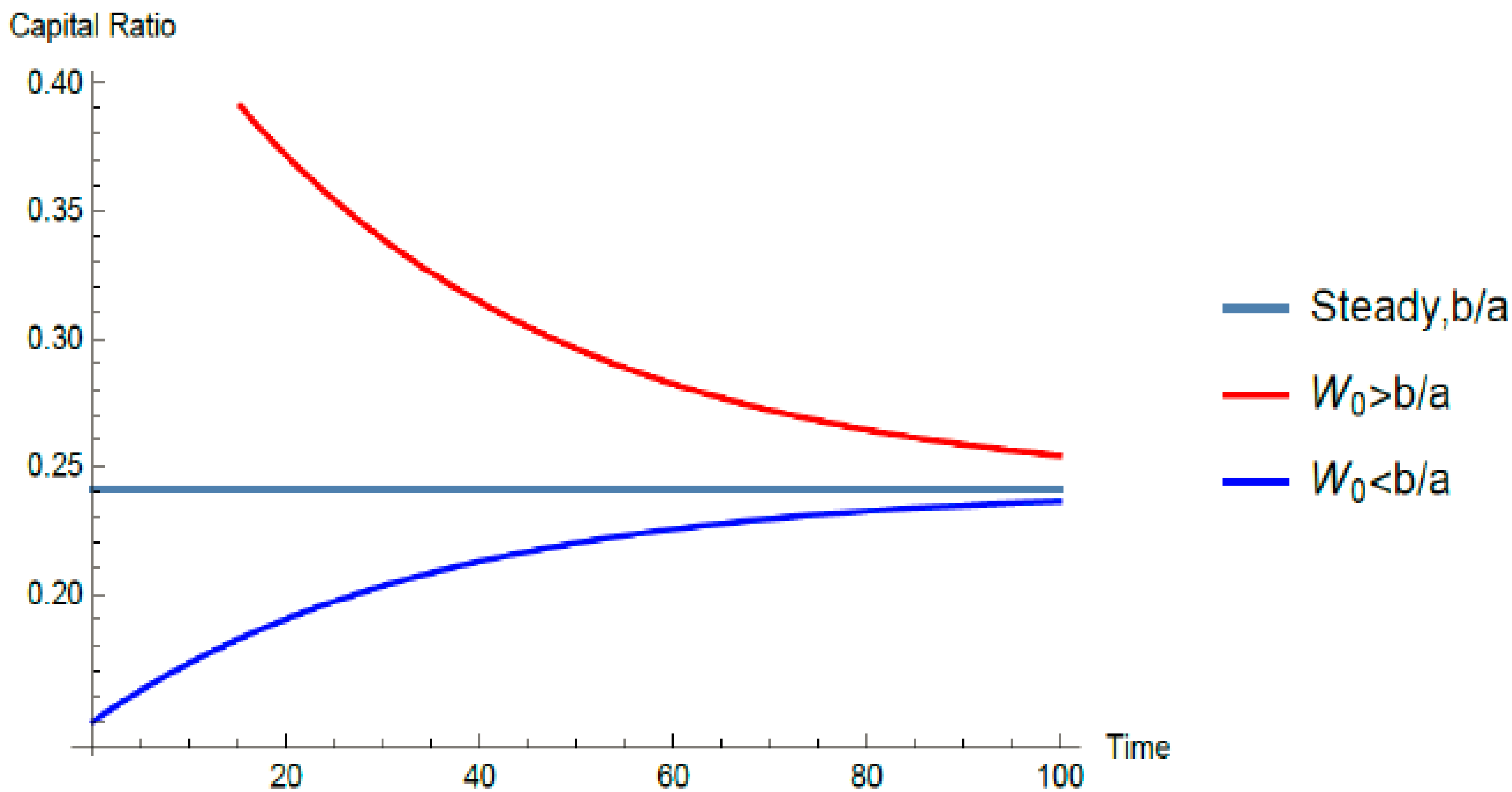
$$dw/dt = -aw(t) + b\ ; \quad 0 < w \le 1\ .$$

where b measures the impact of real market factors.
The solution to ODE is

$$w(t) = Exp[-a\,t]\{w_T + b\int_0^t exp[a\,u]du\}$$

Trajectories for $a_{14} = 0.018;\ b_{14} = 0.004; b/a = 0.222\ .$

Figure II
Trajectories of the capital ratio



The logistic equation for the macro-financial leverage:

$$dl(t) = [al(t) - bl^2(t)]dt \,; \quad 1 \le l < \infty \,.$$

where $a > 0$, and $b > 0$ measures feedbacks in the debt market.

The balanced financial market (BFM) condition:

$$\rho = r + (\mu - r)l \quad \text{for any } 1 \le l < \infty \,.$$

Given parameters r, μ, ρ, the root of (24):

$$K \equiv c / a = a / b$$

is a steady leverage for the logistic model.

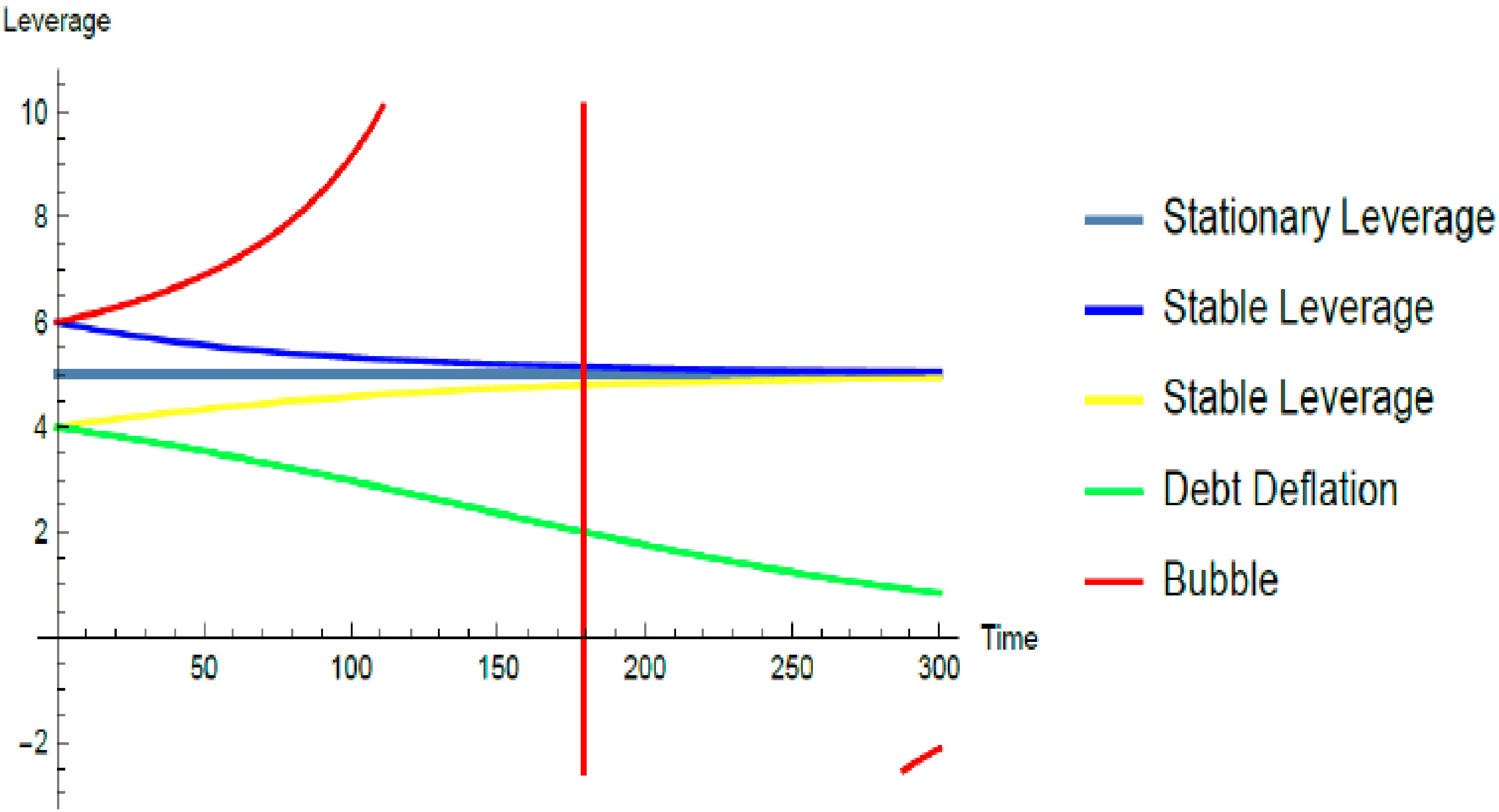
The family of solutions to the logistic equation:

$$l(t) = K\{1 + (\frac{K}{l_0} - 1)\exp[-at]\}^{-1}$$

where l_0 is an initial leverage.

Figure III

Trajectories of the logistic leverage



The long term debt demand and supply

Expected *ROE* depends upon leverage:

$$\rho(l) = r + (\mu - r)l \ .$$

It is an indicator of the long term debt supply because the rate of return on the borrowers' capital is higher, the larger loans they are able to take by leveraging their positions up, and vice versa.

Along the same reasoning the demand for debt:

$$\mu(l^{-1}) = r + (\rho - r)l^{-1} \ ,$$

whereby the knowledge of current rates ρ and r would help creditors to estimate their expected *ROA* on the aggregate portfolio.

The anchor leverage

Competitive adjustments ultimately equalize
Returns on assets and equities:

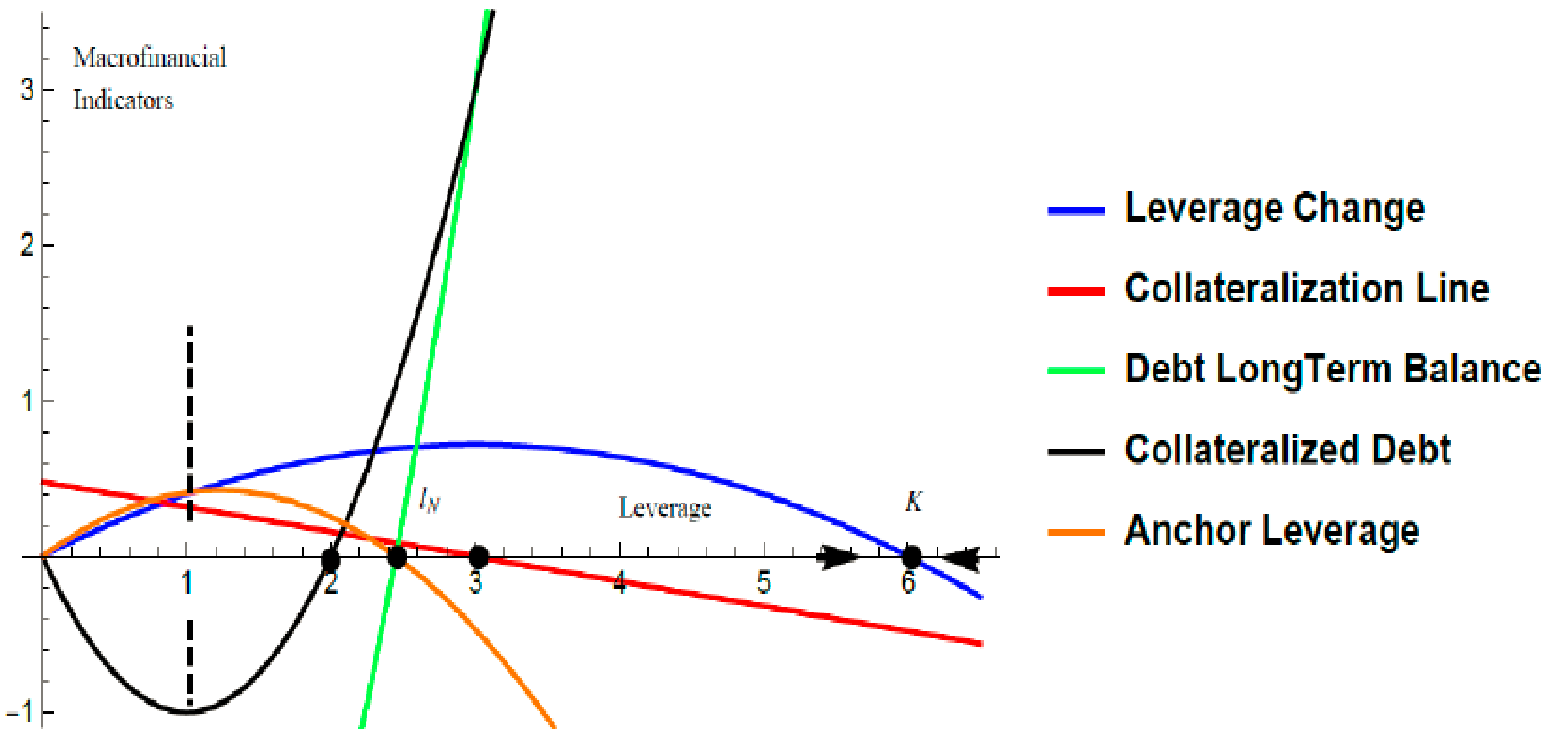
$$\mu(l_N^{-1}) = \rho(l_N)\,.$$

The positive root, l_N, defines the *anchor leverage*:

$$l_N \equiv K^{0.5} = \left(a/b\right)^{0.5}$$

at which indicators of the debt supply and demand are equated.

Figure IV
Characteristic values of leverage



The coupled dynamics of the capital ratio and leverage

Macrofinancial interrelations:

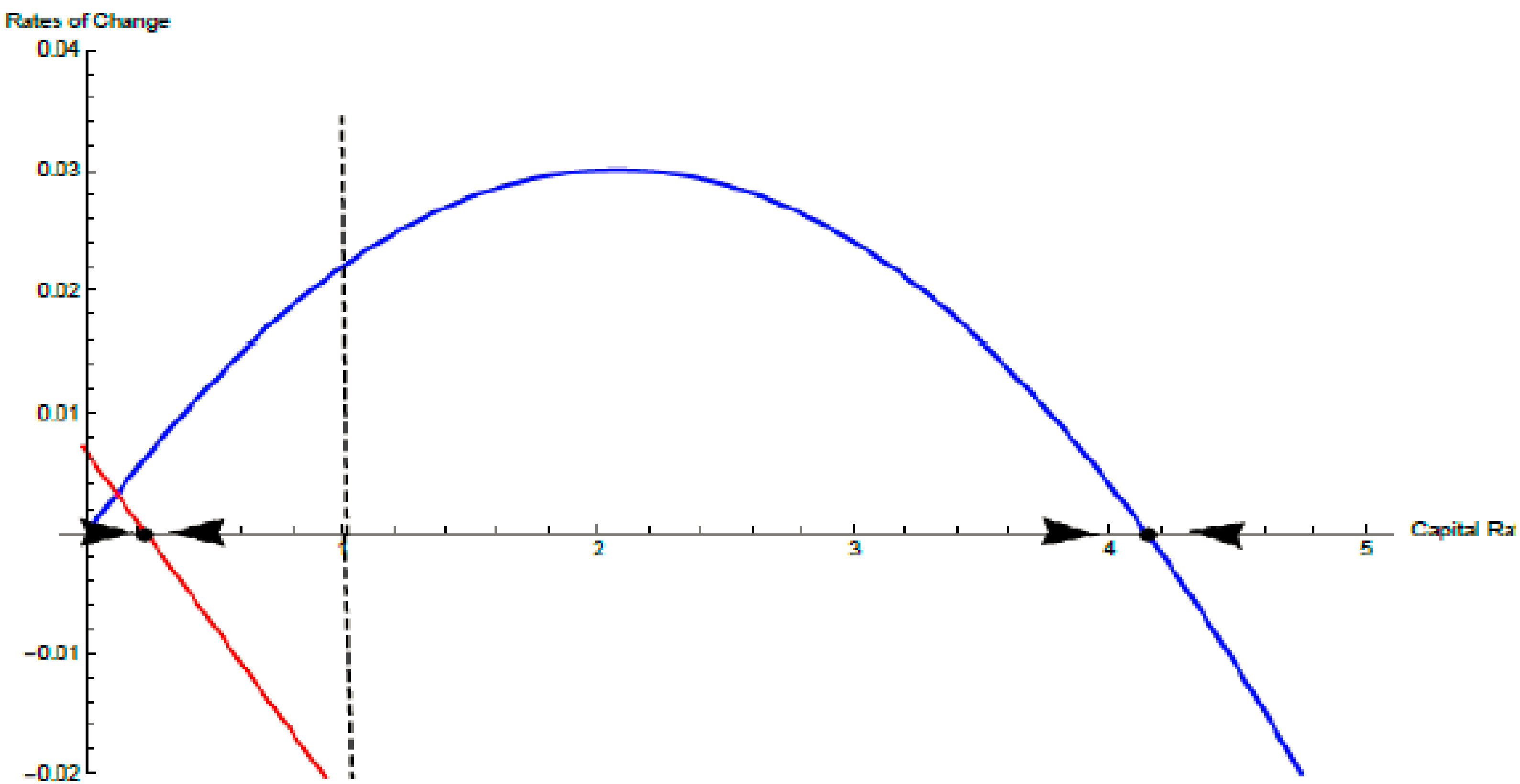
$$dl / dw = -l^2; \ \varepsilon_{lw} = (w / l)(dl / dw) = -1$$

where $\varepsilon_{lw} = -1$ is the unitary elasticity.

The phase portrait with $w_{14} = 0.222; l_{14} = 4.5$.

Figure V

Phase portrait of the coupled capital and debt dynamics



The Ornstein-Uhlenbeck process for the capital ratio

The capital ratio dynamic under uncertainty:

$$dw(t) = [-aw(t) + b]dt + \sigma dB(t)$$

The he standard Brownian motion $B(t) = \int_0^t dB(u)$

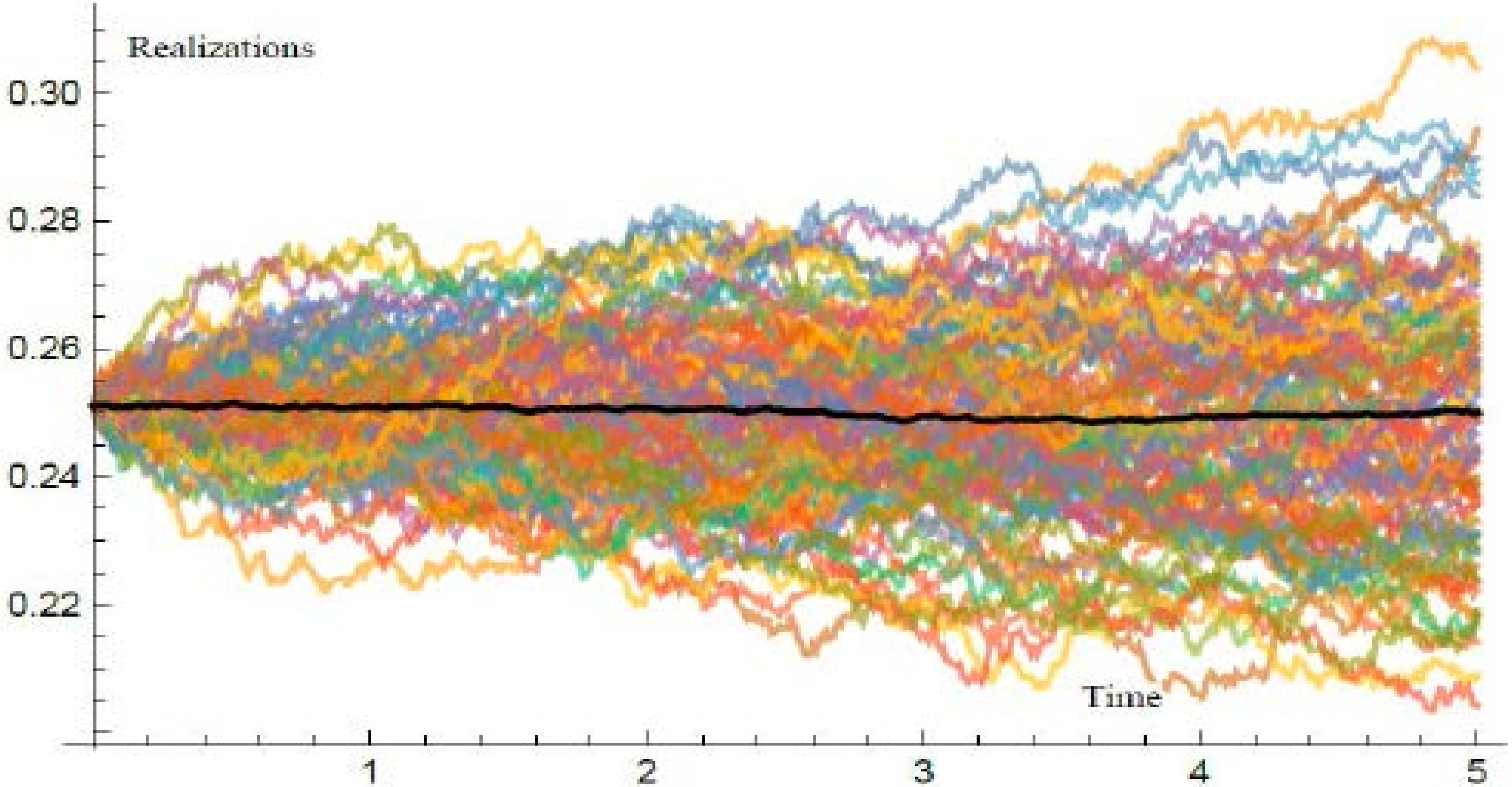
is taken with a market volatility, σ.

The strong solution of the SDE:

$$w(t) = w_T \exp[-at] + \frac{b}{a}(1 - \exp[-at]) + \sigma \int_0^t \exp[a(z-t)]dB(t)$$

Figure VI

Realizations of the stochastic capital ratio



The stationary PDF of the capital ratio

The forward Kolmogorov (or Fokker-Planck) equation:

$$\frac{\partial}{\partial t} p[w(t),t] = -\frac{\partial}{\partial w}\{[-aw(t)+b]p[w(t),t]\} + \frac{1}{2}\frac{\partial^2}{\partial w^2}\{\sigma^2 p[w(t),t]\}$$

with some boundaries and initial conditions.

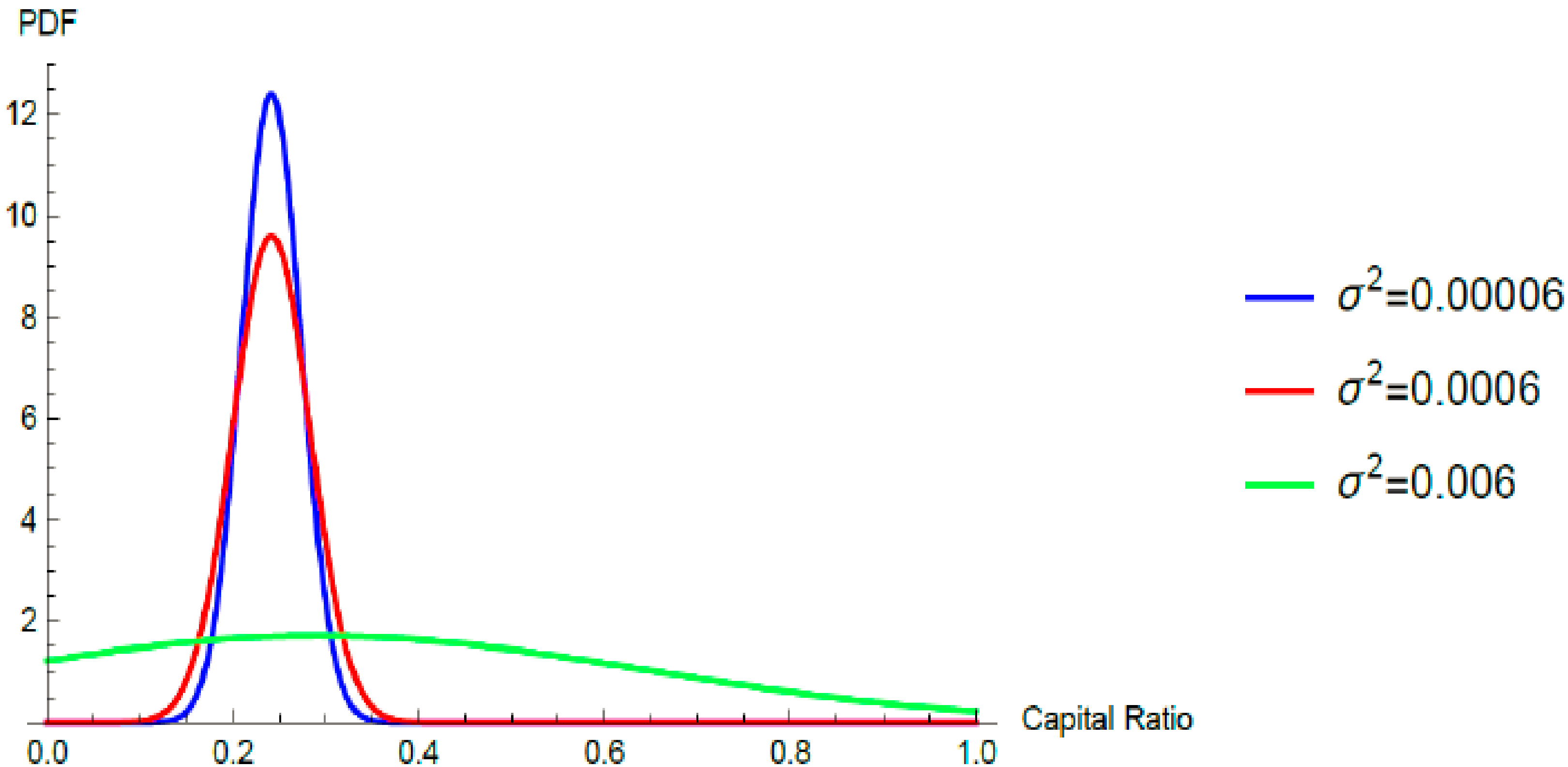
The stationary PDF of the capital ratio:

$$p(w) = \frac{N}{\sigma^2}\exp[-\frac{a}{\sigma^2}(w^2 - 2\frac{b}{a}w)]$$

where N is the constant of normalization.

Figure VII

Stationary distributions of the capital ratio



The logistic diffusion of leverage

Continuous leverage process, $l(t)$, follows the logistic SDE:

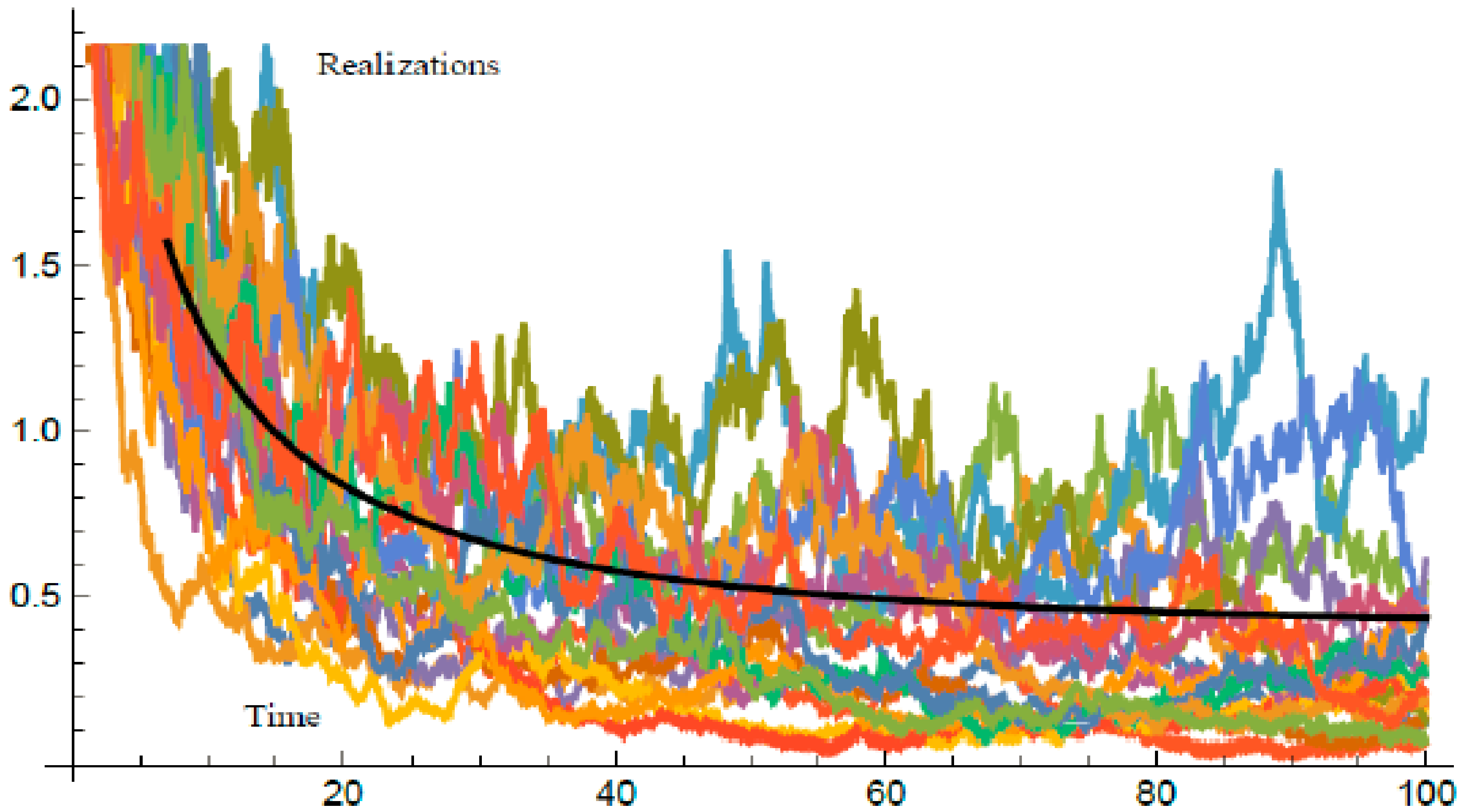
$$dl(t) = [a - bl(t)]l(t)dt + \sigma l(t)dB(t).$$

Its strong solution:

$$l(t) = \frac{l_0 K \exp[(a - 0.5\sigma^2)t + \sigma W(t)]}{K + a l_0 \int_0^t \exp[(a - 0.5\sigma^2)u + \sigma W(u)]du};$$

Figure VIII

Realizations of stochastic leverage



Stationary gamma distribution of leverage

$$-\frac{\partial}{\partial l}[l(a-bl)\,p(l)]+\frac{1}{2}\frac{\partial^2}{\partial l^2}[\sigma^2 l^2 p(l)]=0$$

Stationary probability density function

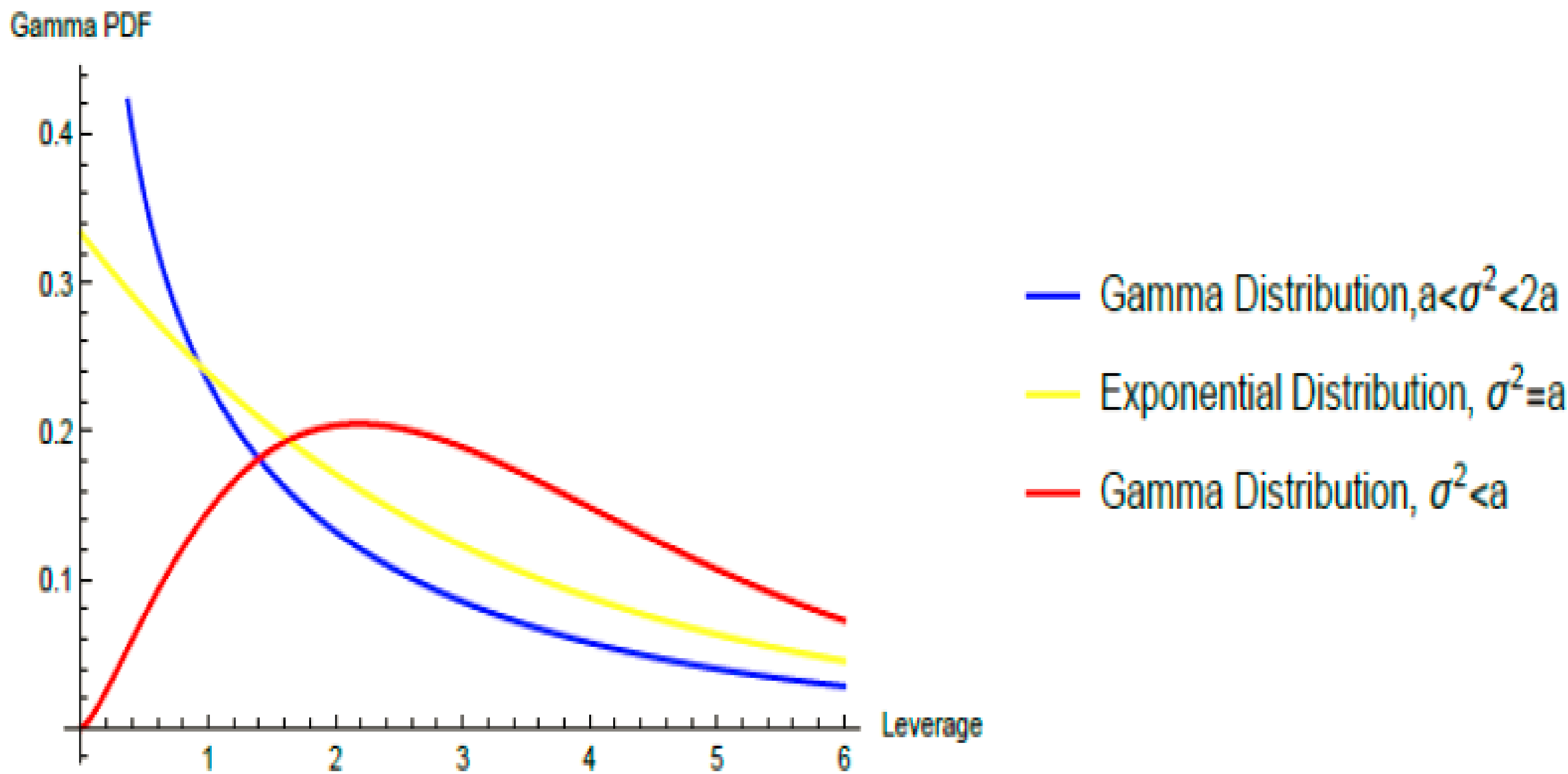
$$p(l;\alpha,\beta)=[\beta^{\alpha}/\Gamma(\alpha)]\,l^{\alpha-1}e^{-\beta l}$$

defines gamma distribution with parameters of shape $\alpha=(2a/\sigma^2)-1$, and rate, $\beta=2b/\sigma^2$, or scale $1/\beta$.

For the positive parameter α (satisfying $0<\sigma^2<2a$) there exist three forms of gamma distribution.

Figure IX

Three forms of a gamma distribution



Characteristics of the gamma disribution

Expectation of the gamma distribution:

$$\langle L \rangle = \alpha / \beta = K - (\sigma^2 / 2b)$$

is larger than its mode:

$$Mode[l] = (\alpha - 1) / \beta = K - \sigma^2 / b\,;$$

the latter does not exist for a "J-shaped" distribution.

Convergence of leverage distributions to the stationary gamma distribution is of special importance.
The long term leverage convergence is measured by a stochastic Lyapunov exponent (SLE):

$$\lambda = \langle a - 2bL \rangle = a - 2b\langle L \rangle = \sigma^2 - a$$

where $\langle . \rangle$ means the ensemble averaging.

The anchor leverage and the mode of stationary gamma distribution

The natural assumption of the global financial system existence implies that its long run leverage has to be associated with the largest (elementary) probability of its realization. As it was shown, financial system exists if the aggregate debt supply equals to its demand which is tantamount to the equality of returns on assets, *ROA*, and on capital, *ROE*. This outcome is expected to take place at the anchor leverage, l_N, in other words, at the most probable value, or the mode, of the random gamma distributed leverage process.
The above said, formally, signifies the equality between the anchor leverage (36) and the mode of the stationary gamma distribution:

$$K^{0.5} = K - (\sigma^2 / b).$$

The critical variance and the random leverage convergence

The binding constraint is satisfied if gamma distributions $p[l(t),t]$ converge to the unimodal stationary gamma distribution which happens for the negative SLE, or for the expected variance no larger than its critical value:

$$\sigma_c^2 = a - \sqrt{ab} \, .$$

It sets the requirement for the implementation of coordinated, comprehensive and consistent reforms of financial markets: such reforms should decrease of the market uncertainty because only convergent random leverage process would stabilize debt markets in the long run.
Successful implementation of financial reforms, by keeping SLE negative, would stabilize leverage in the long run around its most probable value.

Figure 10. Variance and mode of leverage

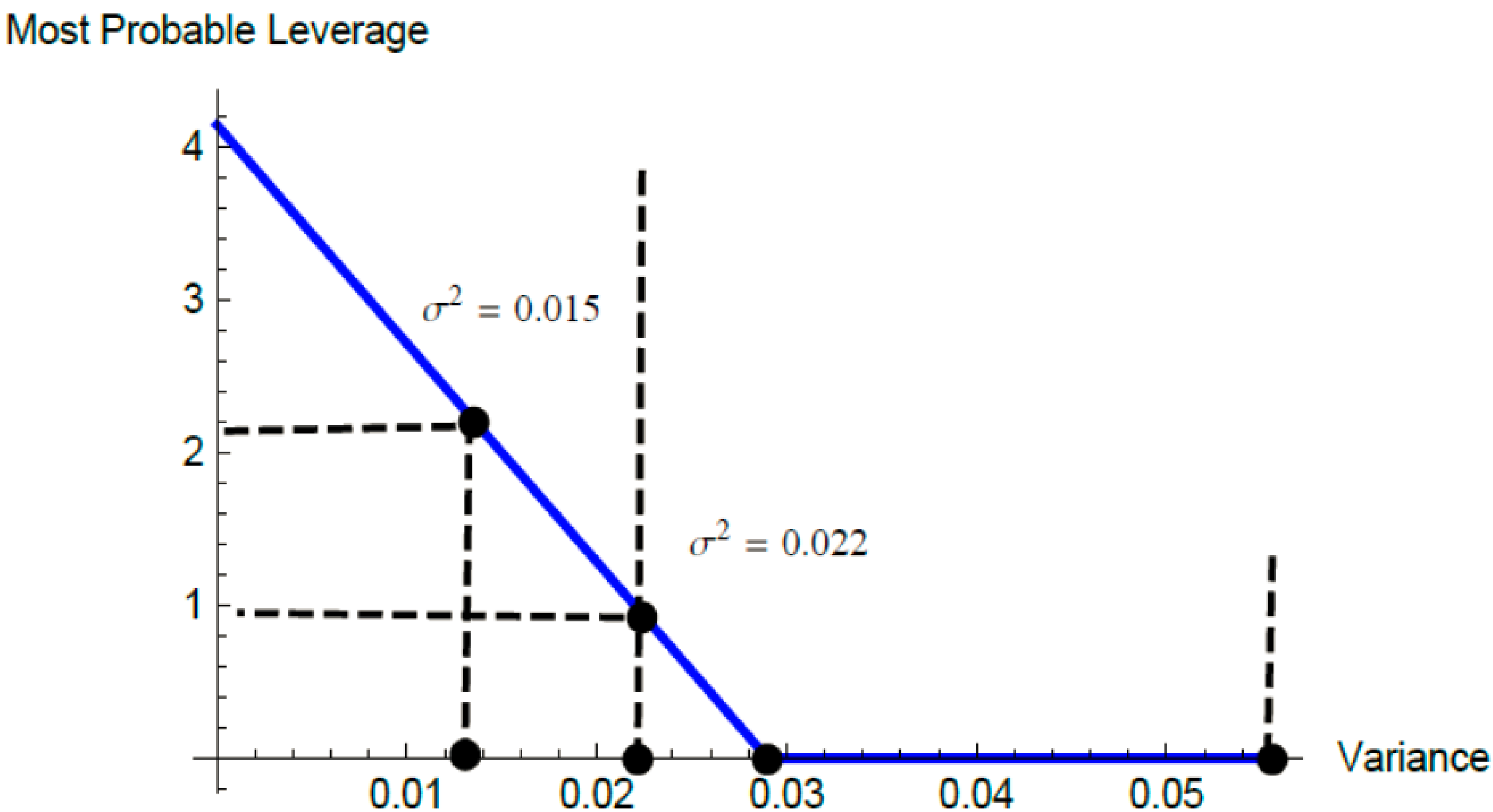


Table II

Leverage variance and mode relations

l^*	α	β	σ^2
a/b	-	-	0
$a/2b$	3	$4b/a$	$a/2$
$(a/b)^{0.5}$	$(a+\sqrt{ab})/(a-\sqrt{ab})$	$2b/(a-\sqrt{ab})$	$a-\sqrt{ab}$
2	$(a+2b)/(a-2b)$	$2b/(a-2b)$	$a-2b$
1	$(a+b)/(a-b)$	$2b/(a-b)$	$a-b$
0	1	$2b/a$	a

The positive effects of the market noise

An interesting phenomenon of certain positive effects of a noisy (risky) debt market. The mode is equal to $l_N = a/b$ for $\sigma^2 = 0$, it decreases to the zero on the interval $0 < \sigma^2 < a$ and stays around zero in the interval $a < \sigma^2 < 2a$. Recall that its meaningful values lie in the interval $0 < \sigma^2 < (a-b)$.

The PDF of leverage simulation

In the PDF simulation the most probable capital ratio was stuck at its deterministic steady state while the mode of gamma distributed leverage, if it existed, was associated with its "anchor" value at which aggregate debt supply and demand were balanced.
Thus, smaller than the steady state anchor leverage in the decoupled stochastic processes facilitated squeezing the bloated debt into a compact and efficient system of financial intermediation.

Deterministic skeletons in stationary KFP equations determine distributions rather than positions of a probabilistic system. Hence the stochastic model does not predict a causal chain of events; instead, particular elementary probabilities of continuous asymptotic capital ratio or leverage are explained by different market configurations associated with them.

The long term debt market configurations

The debt market configurations to the right of the anchor leverage, l_N, would drive leverage down until it comes into the "anchor" position:

$$[\rho(l) - \mu(l)] > 0 \Rightarrow dl / dt < 0 .$$

To the left of the anchor position leverage is smaller, the debt market is in disarray, and the debt demand indicator is higher than that of the debt supply:

$$[\rho(l) - \mu(l)] < 0 \Rightarrow dl / dt > 0 .$$

Table IV

Stationary leverage and its elementary probabilities

Leverage	1	1.5	1.8	2.07	3.0	3.98	6.0
Spread, $(\rho - \mu)$	–0.095	–0.039	–0.017	0	0.046	0.084	0.153
PDF	0.181	0.243	0.258	0.26	0.219	0.148	0.049

Figure XI

The long-term leverage invariant

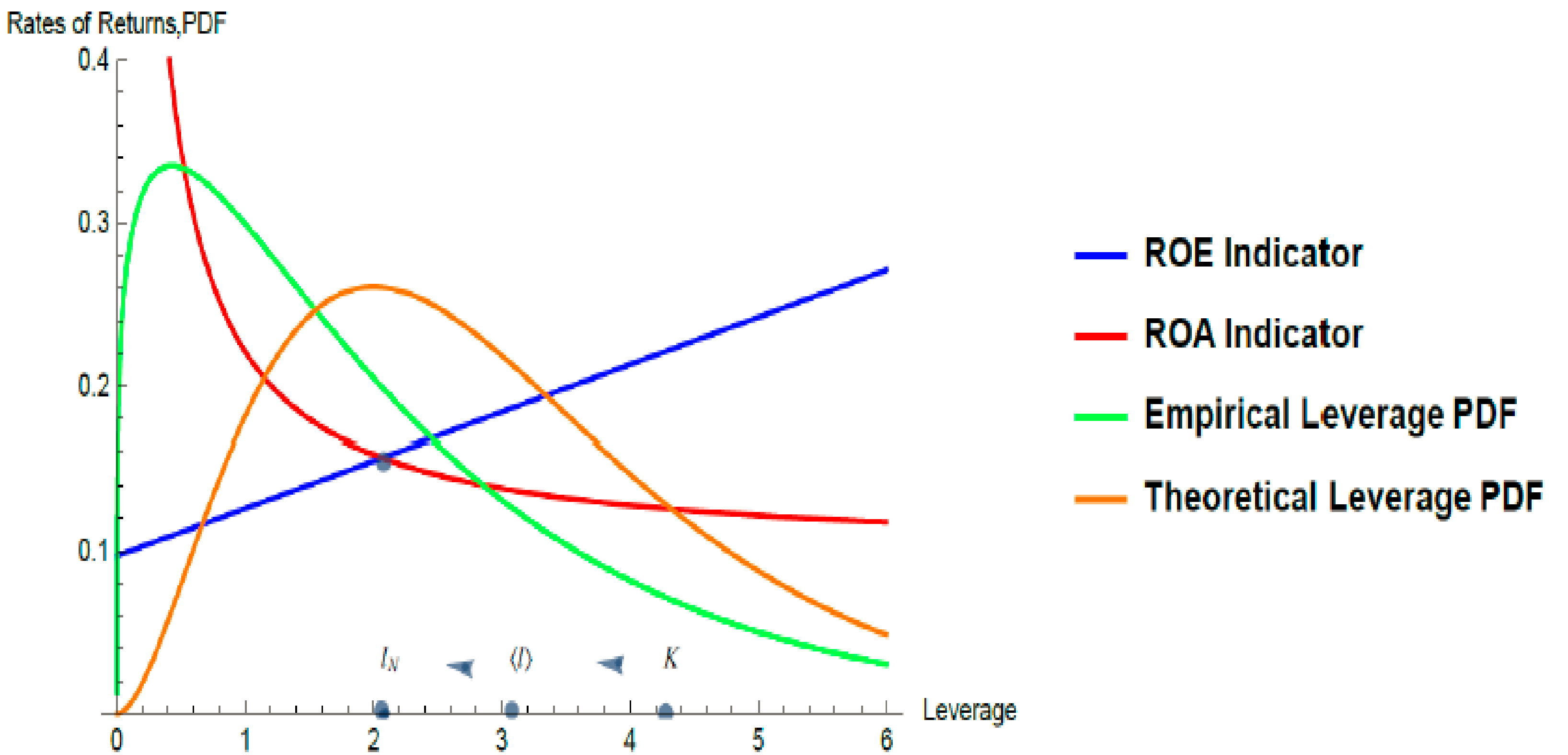


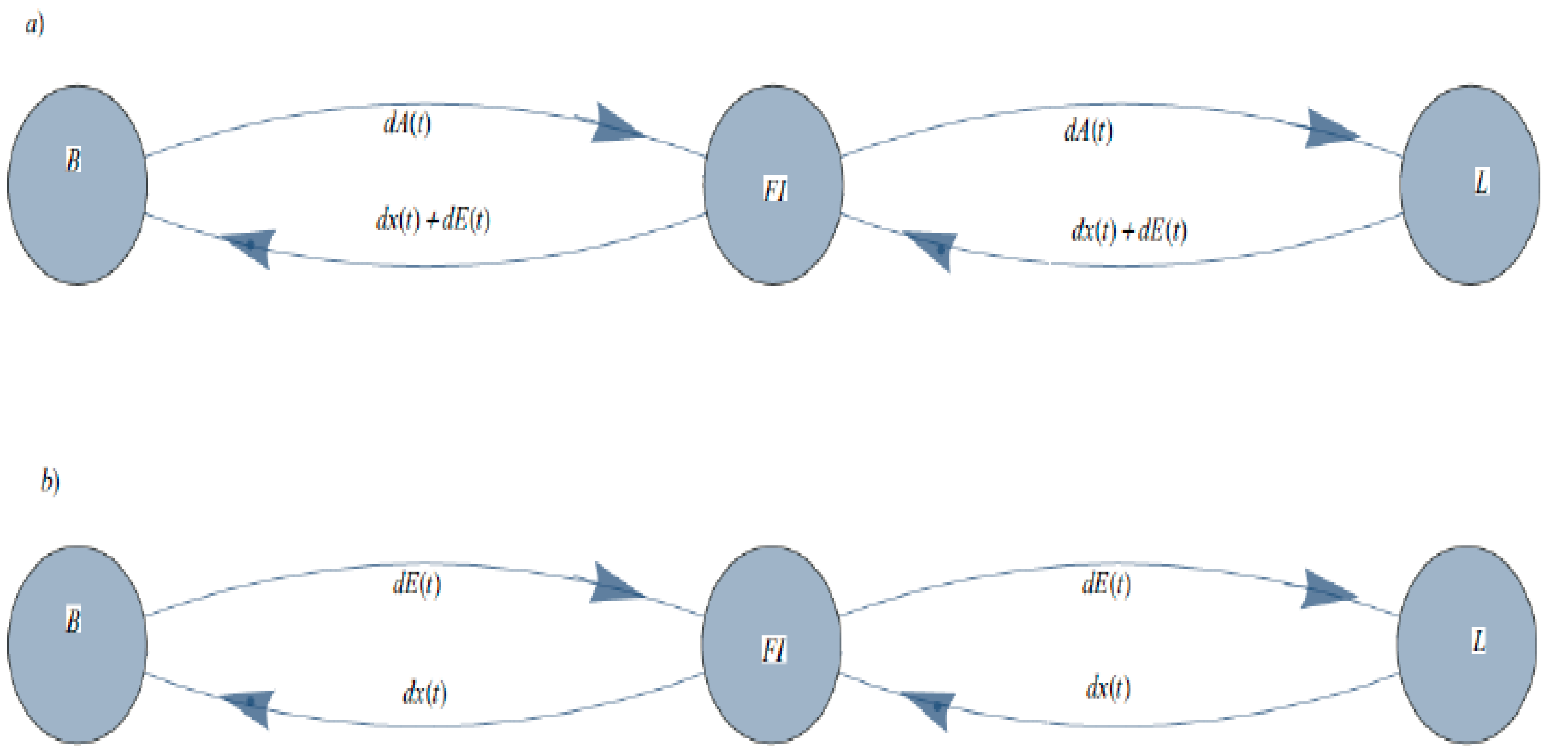
Table V

Scenarios of global assets, debt and capital ($ trillion)

	Assets, A	Capital, E	Debt, x
Year 2017	316.2	79.2	237.0
$w^* l^*$ scenario	315.5	76.2	239.3
$w^* l_N$ scenario,	157.7	76.2	81.5

Fig. XII

Aggregate borrowing and lending (initial and final positions)



A financial intermediary can be represented as

$$\mu l = r(l-1) + \rho ,$$

if normalized by capital, or as

$$\mu \frac{l}{l-1} = r + \frac{1}{l-1} \rho ,$$

if normalized by debt. At the leverage of 2 the equality $dx = dE$ takes place for fungible obligations. Both equations become, $2\mu = r + \rho$, thus revealing no preferences of borrowers over creditors, or vice versa. Hence at $l_2^* = 2$ graphs in **Fig XII** have the same vertices and edges; net positions of the market participants are zero. Their gross notionals are different: Graph **b)** has a total notional $2dA$, (dA - for financial intermediation) which is twice smaller than for the first graph.

Regaining its structure the macrofinancial system operates more effectively.

Somewhat similar processes could be noticed in the segment of the credit default swap (CDS) contracts: their outstanding notional amount was compressed from $61.2 trillion at the end of 2007 to $9.4 trillion in 2018. Thus, the growing fungibility of bonds and negotiable loans via their standardization would drive the leverage to its anchor value in the long run however close to the factor of 2[1].

The natural rate of return (refinancing)

Resolving equations with regard to the rate of refinancing $r \equiv ROR$:

$$\mu(r) = \rho l^{-1} + (1 - l^{-1})r$$
$$\rho(r) = \mu l + (1 - l)r$$

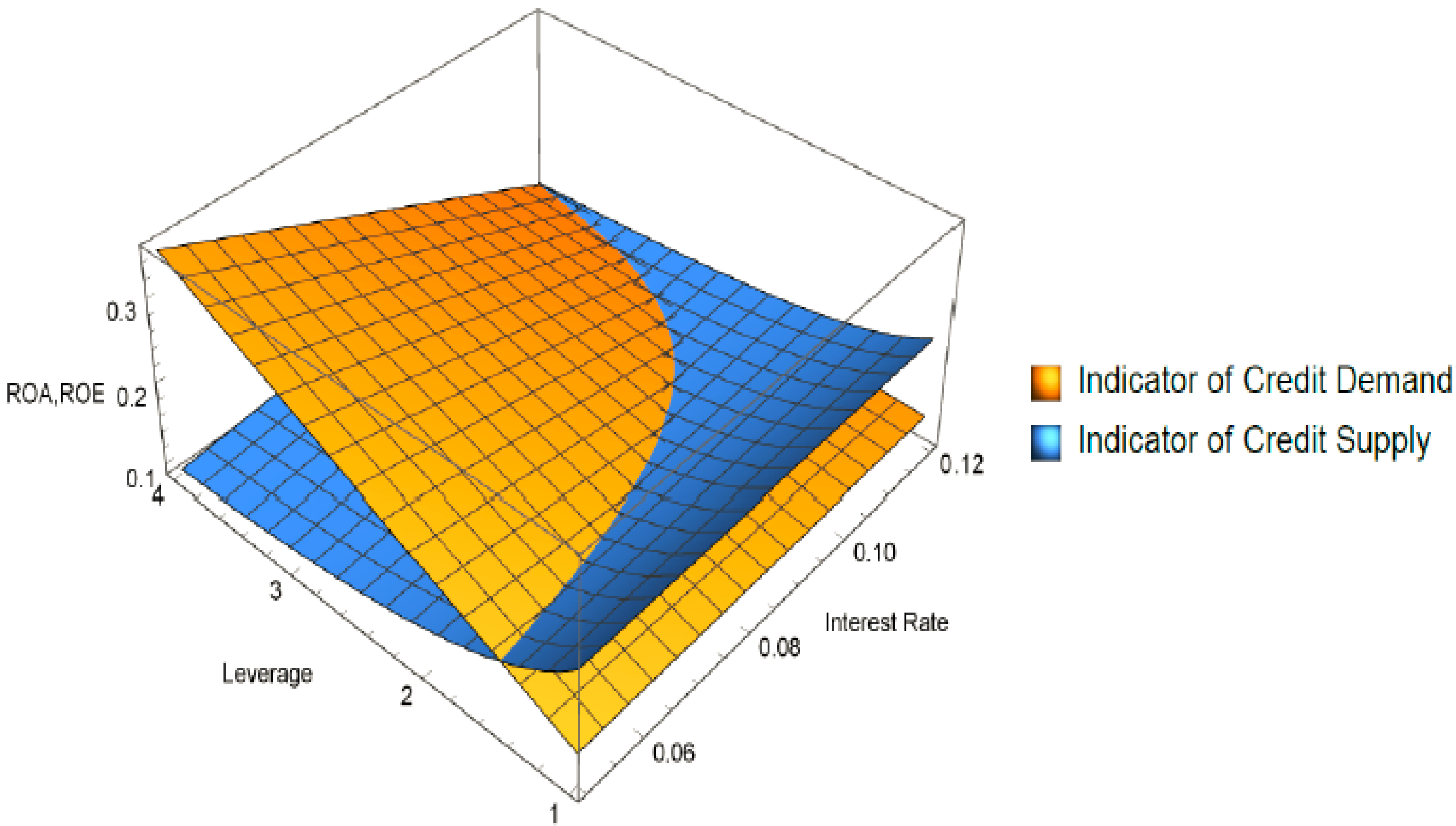
explains the long term behaviour of savers and borrowers coordinated by their rates of return.

Condition : $\mu(r_N) = \rho(r_N)$, gives the "natural" rate of return

$$r_N = (\rho - \mu l^2) / (1 - l^2).$$

Figure XIV

Indicators of credit demand and supply



The alternative noise hypotheses

Alternative processes:

$$dw(t) = [-aw(t) + b]dt + \sigma w(t)dB(t)$$

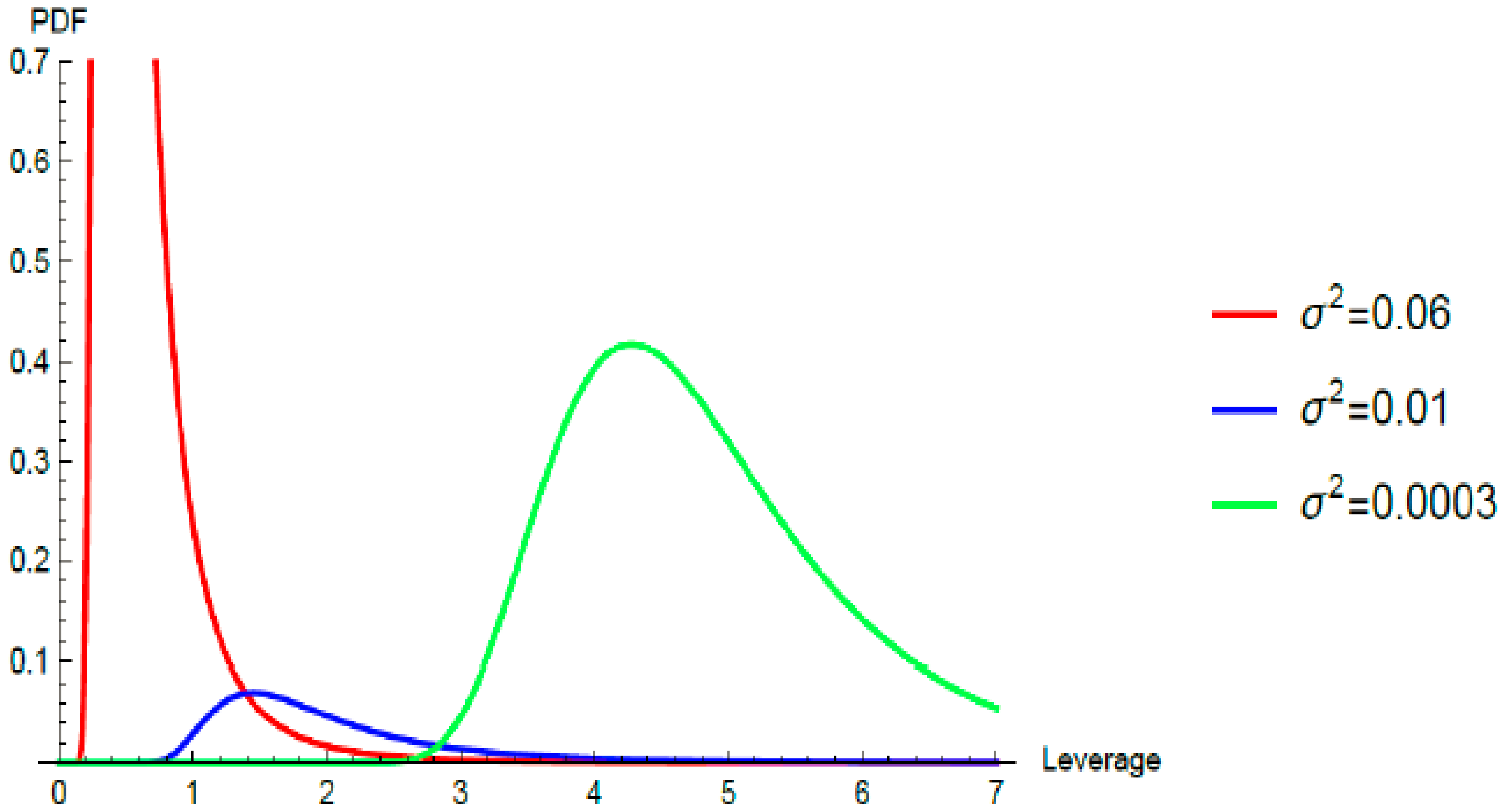
$$dl(t) = [a - bl(t)]\, l(t)dt + \sigma l^2(t)dB(t)$$

The appropriate stationary probability density functions:

$$p(w) = N w^{-\frac{2a}{\sigma^2}-2} \exp[-\frac{2b}{\sigma^2}\frac{1}{w}]; \quad 0 \le w \le 1$$

$$p(l) = \frac{N}{\sigma^2} l^{-4} \exp[-\frac{1}{\sigma^2}(al^{-2} - 2bl^{-1})]; \quad l \ge 1 .$$

Figure XIII
Leverage PDFs of the alternative hypothesis



The model discussion

The alternative model supported the idea of Mao et al (2000) about the random process stabilization that could have been achieved by pumping noise into a system. Yet, in the context of our particular leverage model excessive noise drove the system to its virtual collapse thus realizing *bona fide* a "lethal stabilization".

The gamma distribution does not exist in the interval $a < \sigma^2 < 2a$, and that posits some difficulties with its application; actually inconsistencies could be easily avoided via averaging empirical data over some periods.
The alternative distribution exists without any constraints imposed upon the variance, but its most probable value is meaningless.
So can it be qualified as a substantive improvement of the basic framework, from the economic point of view?