Formal solution of $\hbar-$ KP hierachy

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Tau-function

Hereafter we work with the $\hbar-$ KP hierarchy from K. Takasaki and T. Takebe, *Integrable hierarchies and dispersionless limit*, Rev. Math.Phys. **7** (1995) 743-808. Tau-function of the hierarchy is a function $\tau = \tau(\mathbf{t})$ depended on the infinite set of time variables $\mathbf{t} = \{t_1, t_2, \ldots\}$. There is also contains $\hbar$ as a parameter but we will not write it explicitly. Below we use the notation

$$\tau^{[z_1,\ldots,z_m]}(\mathbf{t}) = e^{\hbar(D(z_1)+\ldots+D(z_m))}\,\tau,$$

where

$$D(z) = \sum_{k\geq 1} \frac{z^{-k}}{k}\,\partial_k; \quad \partial_k = \partial/\partial t_k. \tag{1}$$

Hirota relation

Let $\hbar > 0$. A function τ is called τ-function of $\hbar-$ KP hierarchy if for any z_1, z_2, z_3 it satisfies $\hbar-$Hirota functional relation.

$$(z_1 - z_2)\tau^{[z_1,z_2]}\tau^{[z_3]} + (z_2 - z_3)\tau^{[z_2,z_3]}\tau^{[z_1]} + (z_3 - z_1)\tau^{[z_3,z_1]}\tau^{[z_2]} = 0. \quad (2)$$

For $\hbar = 1$ it gives the ordinary Hirota relation.

Theorem

1. A function τ is τ-function of $\hbar-$ KP hierarchy if and only if for any z_1, z_2

$$\hbar\partial_1 \log \frac{\tau^{[z_1]}}{\tau^{[z_2]}} = (z_2 - z_1)\left(\frac{\tau^{[z_1,z_2]}\tau}{\tau^{[z_1]}\tau^{[z_2]}} - 1\right). \quad (3)$$

2. A function τ is τ-function of $\hbar-$ KP hierarchy if and only if

$$\prod_{1\le i<j\le m} (z_j - z_i) \cdot \tau^{[z_1,\ldots,z_m]}\tau^{m-1} = \det_{1\le j,k\le m}\left((z_j - \hbar\partial_1)^{k-1}\tau^{[z_j]}\right) \quad (4)$$

for any $m \geq 2$ and any $z_1, \ldots, z_m$.

Schur polynomials

For description of τ-function we use Schur polynomials. An elementary Schur polynomial $h_k(\mathbf{t})$ depends from natural number k and infinite number of variable $\mathbf{t} = (t_1, t_2, \dots)$. It is defined by the generating series

$$\exp(\sum_{k\geq 1} t_k z^k) = \sum_{k\geq 0} h_k(\mathbf{t}) z^k.$$

An general Schur polynomial $s_\lambda(t)$ depends from a Young diagram $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_\ell]$ of degree $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_\ell$ with a $\ell = \ell(\lambda) \geq 0$ rows of positive lengths $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0$.

The general Schur polynomial is the determinant

$$s_\lambda(\mathbf{t}) = \det_{i,j=1,\dots,\ell(\lambda)} h_{\lambda_i - i + j}(\mathbf{t}).$$

Formal tau-function

Let us introduce a deformation $\partial_k^\hbar$ of the $\partial_k = \partial/\partial t_k$

$$\partial_k^\hbar = \sum_{l=1}^{k} \frac{\hbar^{l-1} k}{l!} \sum_{\substack{k_1,\ldots,k_l \geq 1 \\ k_1+\ldots+k_l = k}} \frac{\partial_{k_1} \ldots \partial_{k_l}}{k_1 \, \ldots \, k_l} = \partial_k + \hbar \sum_{l=1}^{k-1} \frac{k \partial_l \partial_{k-l}}{2l(k-l)} + O(\hbar^2). \quad (5)$$

In the KP theory the first variable t_1 is distinguished. Thus we will consider τ-function as evolution from a function $\tau(x,0)$ of one variable by $\hbar$- KP flows

$$\tau(x,0) \to \tau(x,\mathbf{t}) = f(x)\hat{\tau}(x+t_1, t_2, t_3, \ldots).$$

A formal τ-function of $\hbar-$ KP hierarchy we mean a formal series for a function $\tau(x,\mathbf{t})$ of this type, that satisfy the Hirota equation by $\mathbf{t}$ for any x.

Our first goal is a construction of formal τ-functions of $\hbar-$ KP hierarchy from f, $\tau(x;\mathbf{0})$, and Cauchy-like data

$$\partial_k^\hbar \tau(x;\mathbf{t}) \Big|_{\mathbf{t}=0} .$$

Theorem

Let $\hbar \neq 0$, and $f(x)$, $c_k(x)$, $k = 0, 2, 3, \ldots$, be arbitrary infinitely differentiable functions of x (with $c_0(x)$ being not identically 0). Put $c_\emptyset(x) = c_0(x)$ and

$$c_\lambda(x) = (c_0(x))^{1-\ell(\lambda)} \det_{1 \leq i,j \leq \ell(\lambda)} \left[\sum_{k=0}^{j-1} (-\hbar)^k C_{j-1}^k \partial_x^k c_{\lambda_i - i + j - k}(x) \right] \tag{6}$$

for Young diagram $\lambda \neq \emptyset$. Then the series

$$\tau(x; \mathbf{t}) = \sum_\lambda c_\lambda(x) s_\lambda(\mathbf{t}/\hbar) \tag{7}$$

is a formal solution to the $\hbar$-KP hierarchy, where

$$\tau(x; \mathbf{0}) = c_0(x), \quad \partial_k^\hbar \tau(x; \mathbf{t}) \Big|_{\mathbf{t}=0} = \frac{k}{\hbar} c_k(x), \quad k \geq 1$$

and $c_1 = \partial_x c_0 - c_0 \partial_x \log f$.

$\hbar-$ KP hierarchy

For many applications in physics and mathematics one needs to deal with logarithm of the tau-function rather than with the tau-function itself. Let us put

$$F(x;\mathbf{t}) = \hbar^2 \log \tau(x;\mathbf{t}). \tag{8}$$

Then the Hirota equations on $\tau(x;\mathbf{t})$ go to $\hbar-$ KP hierarchy on $F(x;\mathbf{t})$. That is

$$e^{\Delta(z_1)\Delta(z_2)F} = 1 - \frac{\Delta(z_1)\partial_x F - \Delta(z_2)\partial_x F}{z_1 - z_2}, \tag{9}$$

where

$$\Delta(z) = \frac{e^{\hbar D(z)} - 1}{\hbar}. \tag{10}$$

For $\hbar = 1$ this is the ordinary KP hierarchy.
For $\hbar = 0$ this is the dispersionless KP hierarchy.

Our goal is a formula, expressing any solution of $\hbar-$ KP hierarchy by *Cauchy-like data* $f_k(x) = \partial_k^{\hbar} F(x; t_1, t_2, \ldots)\Big|_{\mathbf{t}=0}$.

The $\hbar$-KP hierarchy in terms of $\partial_k^\hbar$

Define some combinatorial constants $\tilde{P}_{ij}(s_1, \ldots, s_m)$ as the number of sequences of positive integers $(i_1, \ldots, i_m)$ and $(j_1, \ldots, j_m)$ such that $i_1 + \ldots + i_m = i$, $j_1 + \ldots + j_m = j$ and $s_k = i_k + j_k - 1$. Put

$$P_{ij}(s_1, \ldots, s_m) = \frac{(-1)^{m+1} ij}{m\, s_1 \ldots s_m} \tilde{P}_{ij}(s_1, \ldots, s_m).$$

Theorem

The $\hbar$-KP hierarchy is equivalent to the system of equations

$$\partial_i^\hbar \partial_j^\hbar F = \sum_{m \geq 1} \sum_{\substack{s_1, \ldots, s_m \geq 1 \\ s_1 + \ldots + s_m = i + j - m}} P_{ij}(s_1, \ldots, s_m)\, \partial\partial_{s_1}^\hbar F \ldots \partial\partial_{s_m}^\hbar F, \tag{11}$$

for the function $F = F(x; \boldsymbol{t})$, where $\partial = \partial_1$.

Combinatorial constants

Let $K_l(l^1,\ldots,l^r)$ be the number of partitions of a set of l elements into ordered groups of $l^1,\ldots,l^r>0$ elements.

Define the constants $P^{\hbar}_{i_1\ldots i_k}\left(\begin{array}{c} s_1\ldots s_m \\ l_1\ldots l_m\end{array}\right)$ from integer positive $m,\{i_r\},\{s_r\},\{l_r\}$ by the following recurrence relations:

1) $P^{\hbar}_{i_1,i_2}\left(\begin{array}{c} s_1\ldots s_m \\ 1\ldots 1\end{array}\right)=P_{i_1i_2}(s_1,\ \ldots\ ,s_m)$ and $P^{\hbar}_{i_1,i_2}\left(\begin{array}{c} s_1\ldots s_m \\ l_1\ldots l_m\end{array}\right)=0$, if $\prod\limits_{j=1}^{m} l_j>1$.

2) $P^{\hbar}_{i_1\ldots i_r}\left(\begin{array}{c} x_1\ldots x_v \\ y_1\ldots y_v\end{array}\right)=\sum P^{\hbar}_{i_1\ldots i_{r-1}}\left(\begin{array}{c} s_1\ldots s_m \\ l_1\ldots l_m\end{array}\right)\frac{\hbar^{\nu(k_1,\ldots,k_m)-1}i_r}{[k_1\ldots k_m]}$
$\times K_{l_1}(l_1^1,\ldots,l_1^{n_1})P_{s_1k_1}(s_1^1\ \ldots\ s_{n_1}^1)\ldots K_{l_m}(l_m^1,\ldots,l_m^{n_m})P_{s_mk_m}(s_1^m\ \ldots\ s_{n_m}^m)$,
where $\nu(k_1,\ldots,k_n)$ is the number of positive numbers between k_i and $[k_1,\ldots,k_n)]=\prod\limits_{i=1}^{n}\max\{k_i,1\}$.

The summation is carried over all sets of integer numbers such that

$$(x_1 \dots x_v) = (s_1^1, \dots s_{n_1}^1, s_1^2, \dots, s_{n_2}^2, \dots, s_1^m, \dots, s_{n_m}^m), \quad s_i = \sum_{j=1}^{n_i} s_j^i,$$

$$(y_1 \dots y_v) = (l_1^1 + 1, \dots, l_{n_1}^1 + 1, l_1^2 + 1, \dots l_{n_2}^2 + 1, \dots, l_1^m + 1, \dots, l_{n_m}^m + 1),$$

$$l_i = \sum_{j=1}^{n_i} l_j^i, \sum_{i=1}^{m} (s_i + l_i) = \sum_{j=1}^{r-1} i_j, \quad \sum_{i=1}^{m} k_i = i_r, \quad \sum_{i=1}^{n_j} s_i^j = k_j + s_j.$$

Theorem

The $\hbar$-KP hierarchy is equivalent to the system of equations for $r \geq 2$

$$\partial_{i_1}^{\hbar} \dots \partial_{i_r}^{\hbar} F = \sum_{m \geq 1} \sum_{\substack{s_1 + l_1 + \dots + s_m + l_m \\ = i_1 + \dots + i_r \\ 1 \leq s_i;\ 1 \leq l_i \leq r-1}} P_{i_1 \dots i_r}^{\hbar} \begin{pmatrix} s_1 \dots s_m \\ l_1 \dots l_m \end{pmatrix} \partial^{l_1} \partial_{s_1}^{\hbar} F \dots \partial^{l_m} \partial_{s_m}^{\hbar} F \tag{12}$$

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Operators $\partial^\hbar_\lambda$ and corresponding variables $t^\hbar_\lambda$

A Young diagram $\lambda=[\lambda_1,\lambda_2,\dots,\lambda_\ell]$ with rows of lengths $\lambda_1,\lambda_2,\dots,\lambda_\ell$ is described by $r_i=card\{j|\lambda_j=i\}$. Denote by $\rho(\lambda)=\lambda_1\lambda_2\dots\lambda_\ell$ and $\sigma(\lambda)=\prod\limits_{n\geq 1} r_n!$. We put also $\partial^\hbar_\lambda=\partial^\hbar_{\lambda_1}\partial^\hbar_{\lambda_2},\dots,\partial^\hbar_{\lambda_\ell}$.
Let us consider the basis

$$m_\lambda(x_1,x_2,\dots,x_n)=\frac{1}{(n-\ell(\lambda))!\,\sigma(\lambda)}\sum_{s\in S_n} x_1^{\lambda_{s(1)}}x_2^{\lambda_{s(2)}}\dots x_n^{\lambda_{s(n)}}$$

in the space of symmetrical polynomials from $x_1,x_2,\dots$.
These polynomials are linear combinations from symmetrical polynomials $t_k=\frac{1}{k}\sum_i x_i^k$. This gives new polynomials $m_\lambda(\mathbf{t})$ from $\mathbf{t}=(t_1,t_2,\dots)$.
The first few functions m_λ are:

$$m_{(1)}(\mathbf{t})=t_1,$$
$$m_{(2)}(\mathbf{t})=2t_2,\quad m_{(1^2)}(\mathbf{t})=\tfrac{1}{2}\,t_1^2-t_2,$$
$$m_{(3)}(\mathbf{t})=3t_3,\quad m_{(12)}(\mathbf{t})=2t_2t_1-3t_3,\quad m_{(1^3)}(\mathbf{t})=\tfrac{1}{6}\,t_1^3-t_2t_1+t_3. \tag{13}$$

Let us put

$$t_\lambda^\hbar := \frac{\sigma(\lambda)}{\rho(\lambda)}\, \hbar^{\ell(\lambda)} m_\lambda(\mathbf{t}/\hbar). \tag{14}$$

The first few $t_\lambda^\hbar$ for $(\lambda) = (1^{r_1} 2^{r_2}, \dots)$ are:

$$\begin{aligned} & t_{(1)}^\hbar = t_1, \\ & t_{(2)}^\hbar = t_2, \quad t_{(1^2)}^\hbar = t_1^2 - 2\hbar t_2, \\ & t_{(3)}^\hbar = t_3, \quad t_{(21)}^\hbar = t_2 t_1 - \tfrac{3}{2}\hbar t_3, \quad t_{(1^3)}^\hbar = t_1^3 - 6\hbar t_2 t_1 + 6\hbar^2 t_3. \end{aligned} \tag{15}$$

Theorem

Any formal series $F(\boldsymbol{t}) = F(t_1, t_2, \dots)$ has a representation in form of formal series

$$F(\boldsymbol{t}) = \sum_\lambda \partial_\lambda^\hbar F(\mathbf{t}')\Big|_{\mathbf{t}'=0} \frac{t_\lambda^\hbar}{\sigma(\lambda)}, \tag{16}$$

Construction of formal solutions

Consider now any family of infinitely differentiable functions

$$f^{\hbar}_{[k]}(x) = f^{\hbar}_k(x) \quad (k = 1, 2, \dots).$$

For other Young diagrams λ we put

$$f^{\hbar}_{\lambda}(x) = \sum_{m \geq 1} \sum_{\substack{s_1 + l_1 + \dots + s_m + l_m = |\lambda| \\ 1 \leq s_i;\ 1 \leq l_i \leq \ell(\lambda) - 1}} P^{\hbar}_{\lambda} \begin{pmatrix} s_1 \dots s_m \\ l_1 \dots l_m \end{pmatrix} \partial^{l_1} f_{s_1}(x) \dots \partial^{l_m} f_{s_m}(x), \tag{17}$$

where

$$P^{\hbar}_{\lambda} \begin{pmatrix} s_1 \dots s_m \\ l_1 \dots l_m \end{pmatrix} = P^{\hbar}_{\lambda_1 \dots \lambda_r} \begin{pmatrix} s_1 \dots s_m \\ l_1 \dots l_m \end{pmatrix}$$

for $\lambda = [\lambda_1, \dots, \lambda_\ell]$.

Theorem

For any $\hbar$ *and any family of smooth or formal functions*

$$\boldsymbol{f} = \{f_0(x), f_1(x), f_2(x), \dots\}$$

there exists a unique solution $F(x;\mathbf{t})$ *of the* $\hbar\text{-KP}$ *hierarchy such that*

$$F(x;\mathbf{0}) = f_0(x) \quad \textit{and} \quad \partial_k^{\hbar} F(x; t_1, t_2, \dots)\Big|_{\mathbf{t}=0} = f_k(x).$$

This solution has the form

$$F(x;\mathbf{t}) = f_0(x) + \sum_{|\lambda|\geq 1} \frac{f_\lambda^{\hbar}(x)}{\sigma(\lambda)}\, t_\lambda^{\hbar}. \tag{18}$$