On the boundary between Lorenz attractor and quaisattractor in Shimizu-Morioka system

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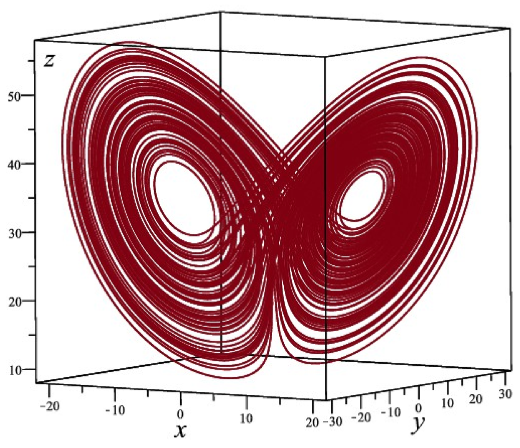
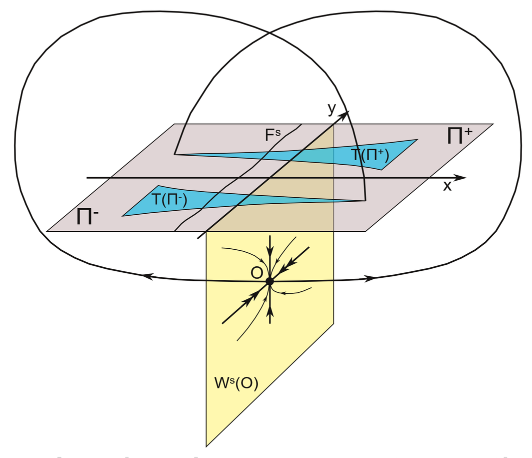
The Lorenz attractor is a strange non-hyperbolic attractor which remains chaotic under small perturbations. For the first time, such chaotic behavior was discovered by E. Lorenz in the following system

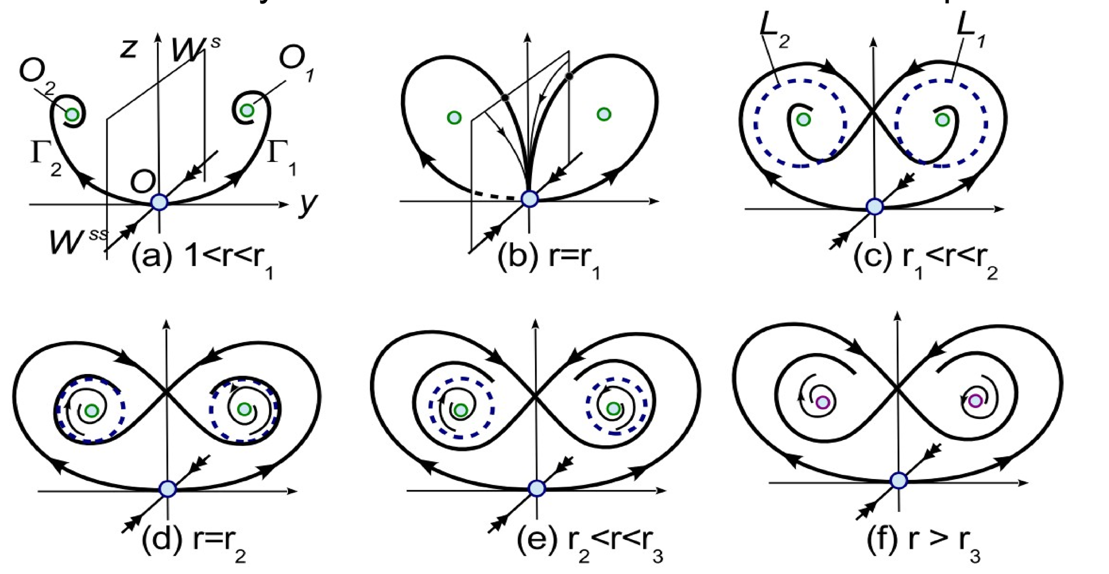


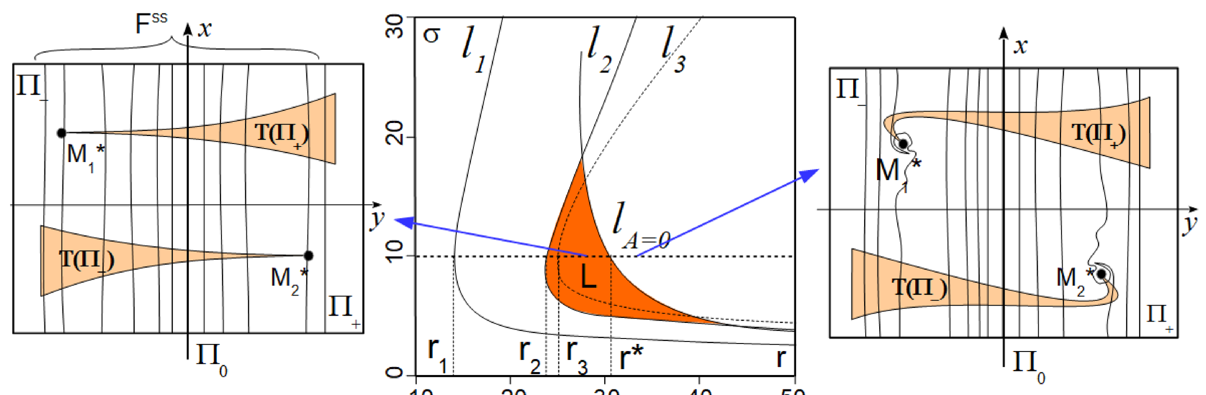
Afraimovich, Bykov, and Shilnikov [1] proposed a geometric model for studying bifurcations and the topological structure of the Lorentz attractor. According to this model, the Lorentz attractor is a stable closed invariant set satisfying certain conditions of pseudo-hyperbolicity.

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = x(r - z) \\ \dot{z} = xy - bz \end{cases}$$

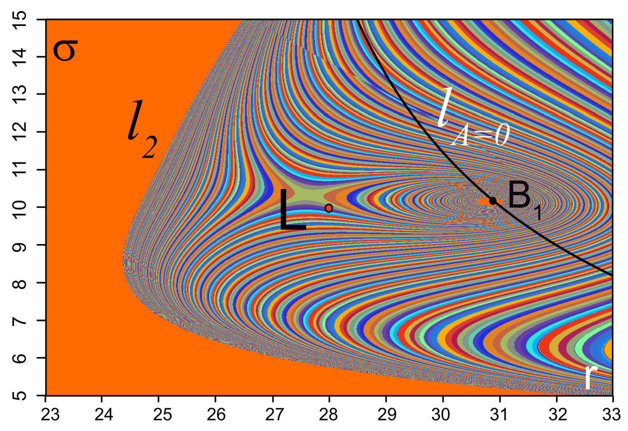


The fulfillment of these conditions provides the existence of a stable foliation for the Poincare map, which allows reducing the problem to the study of a one-dimensional discontinuous map.





In the Lorenz system, the boundary between Lorenz attractor and quasiattractors is formed by the curve $l_{A=0}$ where the separatrix value A of the corresponding Poincare maps vanishes [2]. On the one side from the curve $l_{A=0}$, where A>0, the attractor is pseudo-hyperbolic (PH), and it becomes a quaisattractor (QA) in a sense of Afraivovich and Shilnikov [3] on the other side, when A<0.



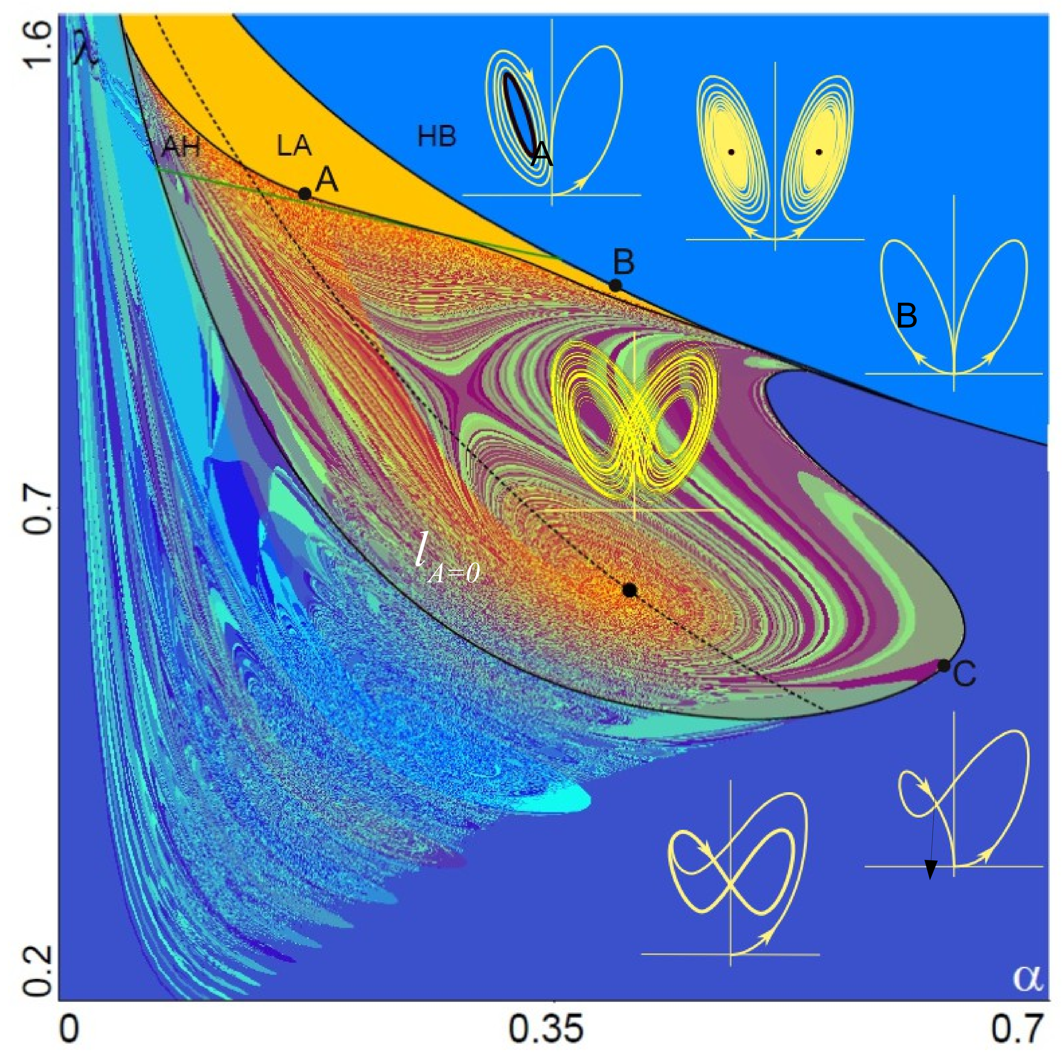
The violating of pseudohyperbolicity on the curve $l_{A=0}$ is associated with the destruction of the stable foliations in the corresponding Poincare map [2].

It is important to note, that in the Lorenz system the saddle index ν of the saddle equilibrium O(0,0,0) is less than 1/2 along the part of the curve $l_{A=0}$

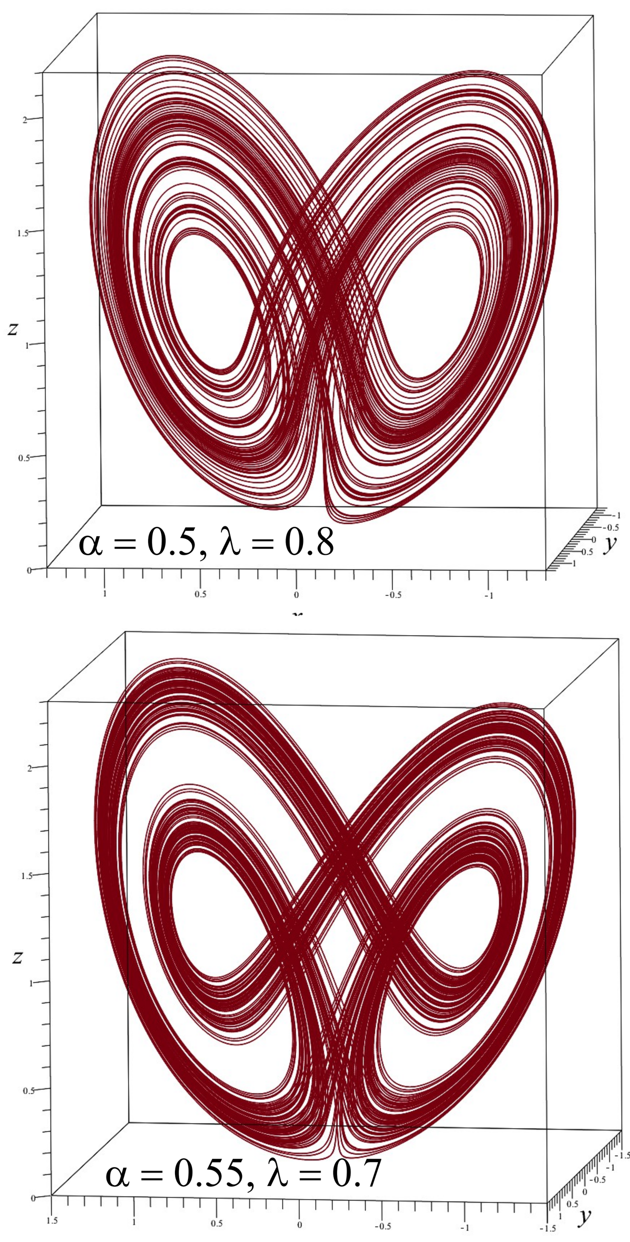
Lorenz attractor in the Shimizu-Morioka system

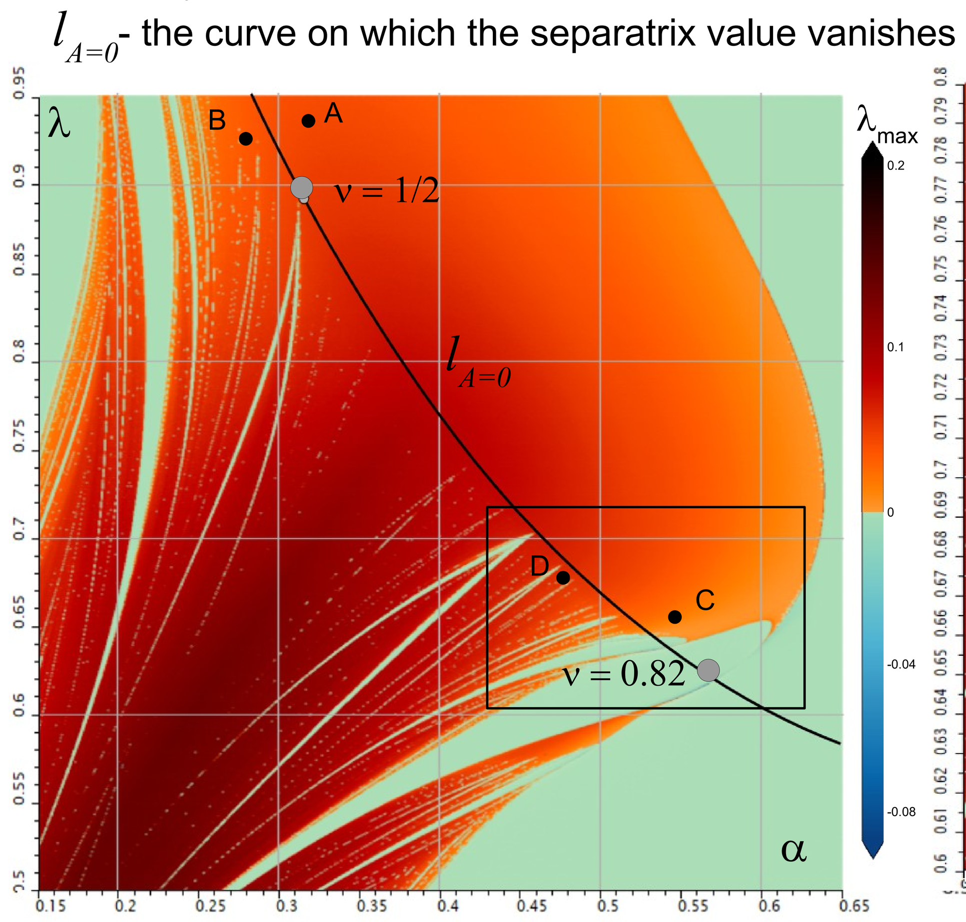
The detailed bifurcation analysis of SM system was done in [4, 5]

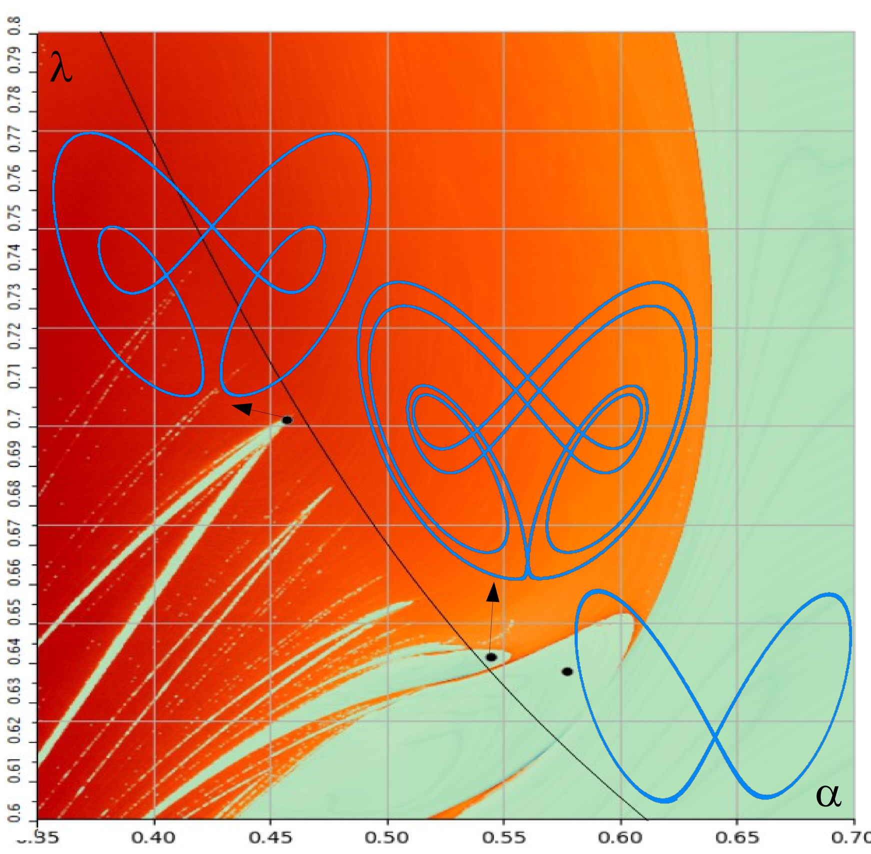
$$\begin{cases} \dot{x} = y, \\ \dot{y} = x - \lambda y - xz, \\ \dot{z} = -\alpha z + x^2. \end{cases}$$

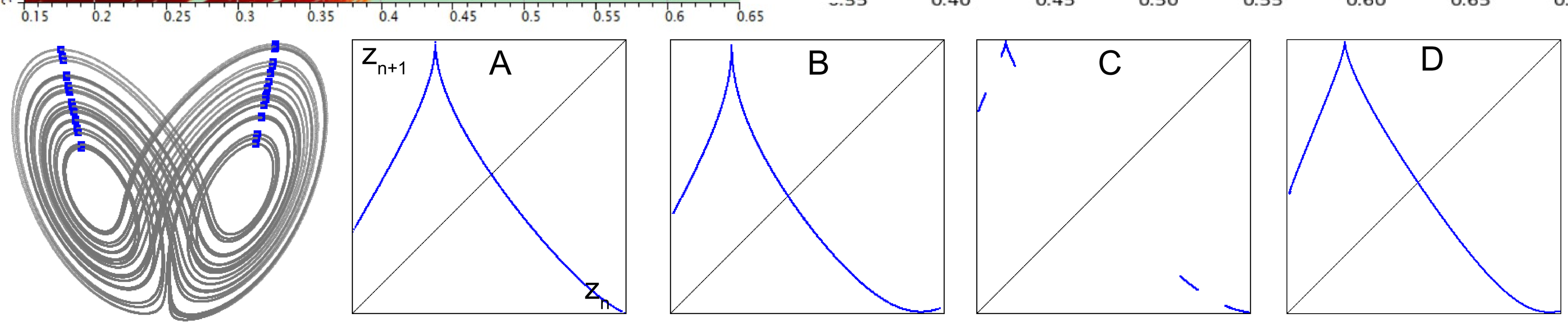


The chart of kneading invariants for the SM-system.
HB — homoclinic butterfly bifurcation
AH — Andronov-Hopf bifurcation
LA — on this curve the unstable separatrix tends to the saddle cycle which is born from HB bifurcation
$l_{A=0}$- the curve on which the separatrix value vanishes



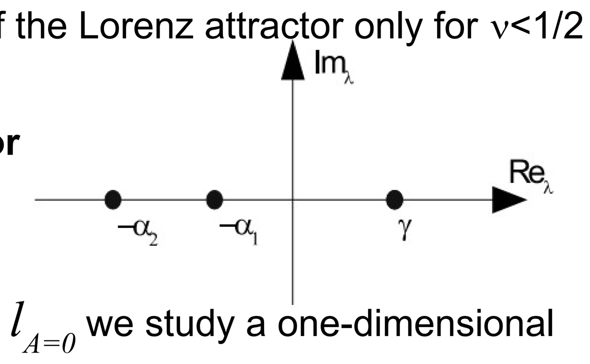






1D maps are constructed by maximal points of z-coordinate of the unstable separatrix

The curve of vanishing of the separatrix value $l_{A=0}$ forms the boundary of the Lorenz attractor only for ν<1/2
ν = α$_1$/γ (For the SM system 0.31 < ν < 0.81)
For ν > 1/2 the boundary between Lorenz attractor and quasiattractor is much more complicated!



For the detailed analysis of bifurcations in the neighborhood of the curve $l_{A=0}$ we study a one-dimensional factor-map of the corresponding Poincare map

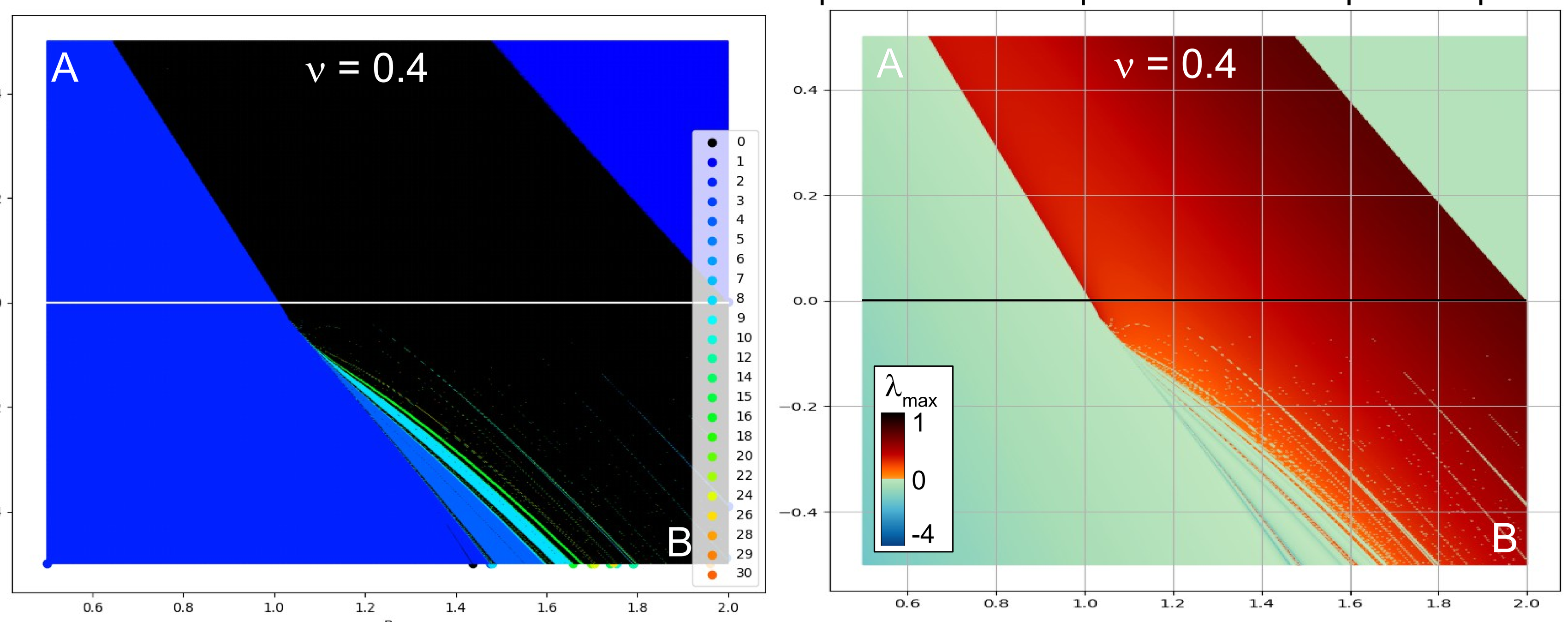
$$\bar{x} = (-1 + \bar{A}|x|^{\nu} + B|x|^{2\nu})sign(x) \quad (*)$$
$$\bar{A} = A\omega^{\nu-1}, \quad B = \omega^{2\nu-1}.$$

Here ω is a parameter of the splitting of a homoclinic loop
A is a sepratrix value.

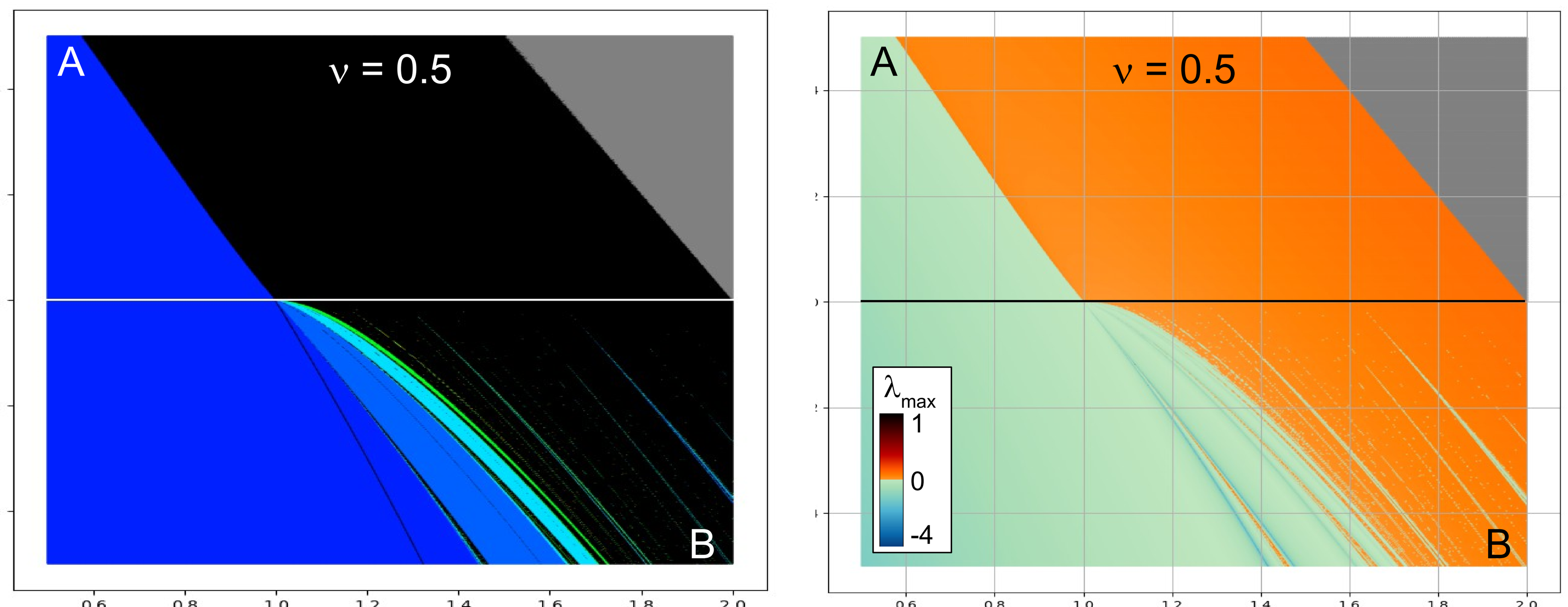
Dynamics of the 1D factor (*) map near the curve $l_{A=0}$

For small values of parameter A, the term $\mathbf{B|x|^{2\nu}}$ in the normal form (*) plays an important role.

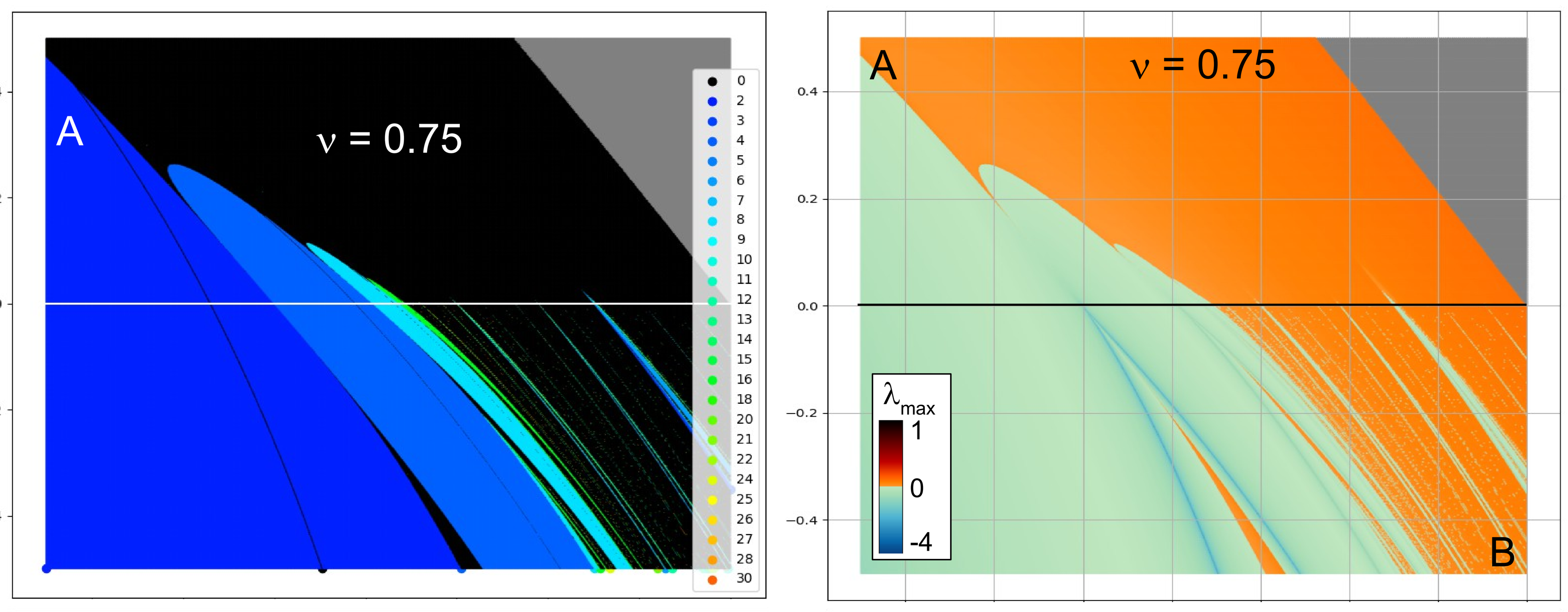
Below, in the charts of periodic regimes (left columns) the black region corresponds to the existence of a nontrivial attractor in the map (*). This attractor is a union of a finite number of intervals. The stability windows are shown in colors. Different colors correspond to different periods of stable periodic points.

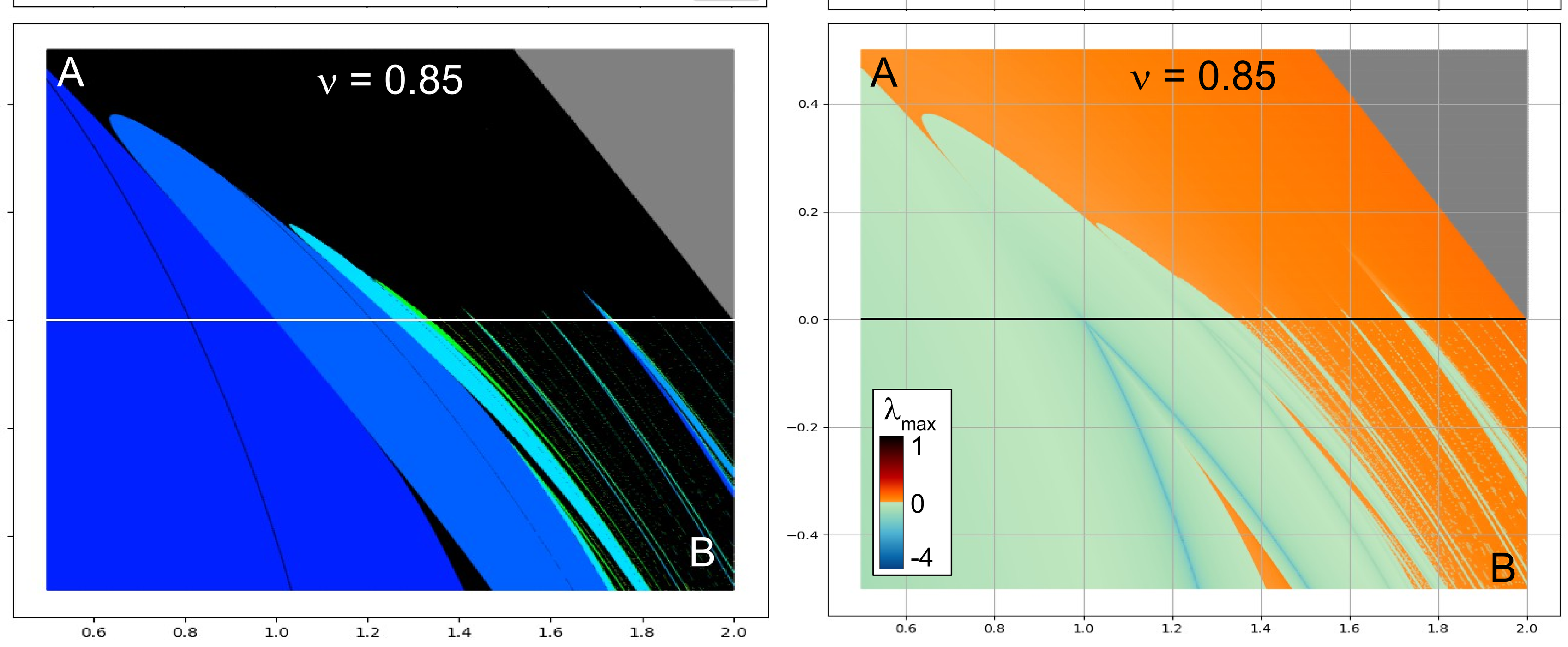


For ν < 1/2, the exponent in term $\mathbf{B|x|^{2\nu}}$ is less than 1; therefore, the one-dimensional map (*) does not have stable periodic orbits at A > 0. The attractor becomes quasiattractor at A < 0.



For ν = 1/2 the stability windows tangent the curve A = 0





For ν > 1/2 the map (*) has a zero derivative at the discontinuity point 0 and, therefore, can possess stable periodic orbits which also exist for positive values of A. Thus, the attractor can become a quasiattractor for A > 0.

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