# On loop corrections to integrable $2D$ sigma model backgrounds

Based on M. Alfimov and A. Litvinov, JHEP**01**(2022)043 and work in progress

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Yerevan, Armenia, July 14, 2022

# Motivation

- The $\beta$-functions in QFT are known to depend on the renormalization scheme.
- In QFT's with one coupling constant we can make the $\beta$-function 2-loop exact (for example, in $\varphi^4$ theory).
- In QFT's with two or more couplings it is not known in general, whether and how it is possible to achieve such a simple form.
- It is particularly interesting to study integrable deformations of 2-dimensional sigma models, for example, $\eta$-deformed $O(N)$ ones with two couplings.
- We know the $\beta$-function for 2-dimensional sigma models up to 4-loop order and how it varies under scheme changes.
- For $D = 2$ target space there are much less different tensor structures and we have a hope to obtain a particularly simple expression for the $\beta$-function in some scheme.
- We know some conjectured all-loop metrics in a certain scheme for $\eta$- and 2-loop ones for $\lambda$-deformed models (Hoare et al.'19), so it could be possible to find a simple expression for higher-loop $\beta$-functions.

# $\beta$-function in the $\varphi^4$ theory

- Consider the $\phi^4$ scalar QFT
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2} m^2 \varphi^2 - \frac{g}{4!}\varphi^4 \ .$$
- $\beta$-function in $4 - \epsilon$ dimensions ('t Hooft'72) is known up to 6-loop order (Kompaniets et al.'16) in the minimal subtraction scheme ('t Hooft'73). The expression at the 2-loop order
$$\dot{g} = -\beta(g), \quad \beta(g) = -\epsilon g + 3g^2 - \frac{17g^3}{3} + \mathcal{O}(g^4) \ .$$
- Change of regularization scheme can be effectively understood as a change of the coupling
$$g \to \tilde{g}(g) = g + \xi_1 g^2 + \xi_2 g^3 + \mathcal{O}(g^4) \ .$$
- The $\beta$-function transforms as the vector field
$$\dot{\tilde{g}} = \frac{\partial \tilde{g}}{\partial g}\dot{g} \to \tilde{\beta}(\tilde{g}) = \left(\frac{\partial \tilde{g}(g)}{\partial g}\right)^{-1} \beta(\tilde{g}(g)) \ ,$$
  where
$$\dot{g} = \frac{dg}{dt}$$
  and $t$ is the logarithm of the renormalization scale.

# Normal form of the $\beta$-function in the $\varphi^4$ theory

- For the $\beta$-function, which starts from $g^2$, corresponding to marginal perturbation, one has

$$\beta(g) = A_1 g^2 + A_2 g^3 + A_3 g^4 + \mathcal{O}(g^5) \to$$
$$\to \tilde{\beta}(\tilde{g}) = A_1 \tilde{g}^2 + A_2 \tilde{g}^3 + (A_3 + A_2 \xi_1 + A_1(\xi_1^2 - \xi_2))\tilde{g}^4 + \mathcal{O}(\tilde{g}^5)\ .$$

- The first two coefficients are scheme independent.

- By choosing

$$\xi_1 = \frac{A_3}{A_2}\ , \quad \xi_2 = \xi_1^2$$

  we can make the $\beta$-function at the 3rd order to be $0$.

- Tuning the renormalization parameters $\xi_k$, one can always find the *normal form* of the $\beta$-function, i.e. the scheme, in which the $\beta$-function is 2-loop exact

$$\tilde{\beta}(\tilde{g}) = A_1 \tilde{g}^2 + A_2 \tilde{g}^3\ .$$

# Sigma models in 2 dimensions

- We study 2-dimensional sigma models

$$S[\boldsymbol{X}] = \frac{1}{4\pi} \int G_{ij}(\boldsymbol{X}) \partial_a X^i \partial_a X^j \, d^2\sigma \; .$$

- The metric $G_{ij}(\boldsymbol{X})$ also depends on some parameters treated as coupling constants, which vary with the scale according to RG flow equation

$$\dot{G}_{ij} + \nabla_i V_j + \nabla_j V_i = -\beta_{ij}(G) \; .$$

- The metric $\beta$-function $\beta_{ij}(G)$ admits the covariant loop expansion

$$\beta_{ij}(G) = \beta^{(1)}_{ij}(G) + \beta^{(2)}_{ij}(G) + \beta^{(3)}_{ij}(G) + \dots \; ,$$

  where $L$-th loop order $\beta$-function coefficient $\beta^{L}_{ij}$ belongs to the finite dimensional space of tensors with given scaling properties.

- It is convenient to have in mind that the metric is proportional to the inverse of the Planck constant, which implies the following scaling for basic tensors

$$G_{ij} \sim \hbar^{-1} \to G^{ij} \sim \hbar \; , \; \Gamma^k_{ij} \sim \hbar^0 \; , \; \nabla_i \sim \hbar^0 \; , \; R_{ijk}{}^{l} \sim \hbar^0 \; , \; R_{ij} \sim \hbar^0 \; , \; R \sim \hbar \; .$$

## $\beta$-function of 2-dimensional sigma model

- The 1-loop $\beta$-function is proportional to the Ricci curvature

$$\beta_{ij}^{(1)} = R_{ij} \ ,$$

and the corresponding RG equation is the celebrated Ricci flow equation.

- Higher loop coefficients $\beta_{ij}^{(L)}$ have been calculated in the minimal subtraction scheme: in 2 loops in (Friedan'80)

$$\beta_{ij}^{(2)} = \frac{1}{2} R_{iklm} R_j{}^{klm} \ .$$

- in 3 loops in (Graham'87, Foakes, Mohammedi'87)

$$\beta_{ij}^{(3)} = \frac{1}{8} \nabla_k R_{ilmn} \nabla^k R_j{}^{lmn} - \frac{1}{16} \nabla_i R_{klmn} \nabla_j R^{klmn} - \\ - \frac{1}{2} R_{imnk} R_{jpq}{}^k R^{mqnp} - \frac{3}{8} R_{iklj} R^{kmnp} R^l{}_{mnp} \ .$$

- Also the 4-loop result has been obtained in (Jack et al.'89).
- The higher loop coefficients $\beta_{ij}^{(L)}$ for $L > 1$ are scheme dependent. They are related by covariant metric redefinitions

$$G_{ij} \to \tilde{G}_{ij} = G_{ij} + \sum_{k=0}^{\infty} G_{ij}^{(k)} \ ,$$

where $G_{ij}^{(k)}$ is of the order $\hbar^k$.

## $\beta$-function for $D = 2$ sigma models

- The $\beta$-function for the SM with *two-dimensional* target space is significantly simplified

$$\beta_{ij}^{(1)} = \frac{1}{2} R G_{ij} \ ,$$
$$\beta_{ij}^{(2)} = \frac{1}{4} R^2 G_{ij} \ ,$$
$$\beta_{ij}^{(3)} = \left( \frac{5}{32} R^3 + \frac{1}{16} (\nabla R)^2 \right) G_{ij} - \frac{1}{16} \nabla_i R \nabla_j R \ ,$$
$$\beta_{ij}^{(4)} = \left( \frac{23}{192} R^4 + \frac{2+\zeta(3)}{32} R^2 \nabla^2 R + \frac{41+12\zeta(3)}{192} R (\nabla R)^2 + \frac{1}{192} (\nabla^2 R)^2 + \right.$$
$$\left. + \frac{1}{192} (\nabla_i \nabla_j R)^2 \right) G_{ij} - \frac{\zeta(3)}{48} R^2 \nabla_i \nabla_j R - \frac{25 + 8\zeta(3)}{192} R \nabla_i R \nabla_j R - \frac{1}{96} (\nabla^2 R) \nabla_i \nabla_j R \ .$$

- Covariant metric redefinition is determined by several tensor structures at every order of $\hbar$

$$G_{ij}^{(0)} = c_1 R G_{ij} \ ,$$
$$G_{ij}^{(1)} = \left( c_2 R^2 + c_3 \nabla^2 R \right) G_{ij} + c_4 \nabla_i \nabla_j R \ ,$$
$$G_{ij}^{(2)} = \left( c_5 R^3 + c_6 (\nabla R)^2 + c_7 R \nabla^2 R + c_8 \nabla^2 \nabla^2 R \right) G_{ij} +$$
$$+ c_9 \nabla_i R \nabla_j R + c_{10} R \nabla_i \nabla_j R + c_{11} \nabla_i \nabla_j \nabla^2 R$$

and so on.

# $\beta$-function for $D = 2$ sigma models in different schemes I

- Up to a diffeomorphism transformation after covariant redefinition metric satisfies the RG flow equation

$$\dot{\tilde{G}}_{ij} + \tilde{\nabla}_i \tilde{V}_j + \tilde{\nabla}_j \tilde{V}_i = -\tilde{\beta}_{ij}(\tilde{G})$$

with some vector field $\tilde{V}_i$ and transformed $\tilde{\beta}_{ij}$.

- 1-loop $\beta$-function $\beta^{(1)}_{ij}$ is obviously scheme-independent.
- Perturbation of the left hand side

$$\begin{aligned}\dot{\tilde{G}}_{ij} &= \dot{G}_{ij} + \dot{G}^{(0)}_{ij} + \ldots = \dot{G}_{ij} + c_1(\dot{R}G_{ij} + R\dot{G}_{ij}) + \ldots = \\ &= \dot{G}_{ij} + \frac{c_1}{2}\Delta R G_{ij} + \ldots = -\frac{1}{2}RG_{ij} + \left(-\frac{1}{4}R^2 + \frac{c_1}{2}\Delta R\right)G_{ij} + \ldots\end{aligned}$$

- Right hand side takes the form

$$\begin{aligned}\tilde{\beta}_{ij}(\tilde{G}) &= \frac{1}{2}\tilde{R}\tilde{G}_{ij} + \left(\tilde{b}^{(2)}_1 \tilde{R}^2 + \tilde{b}^{(2)}_2 \tilde{\Delta}\tilde{R}\right)\tilde{G}_{ij} + \tilde{b}^{(2)}_3 \tilde{\nabla}_i \tilde{\nabla}_j \tilde{R} + \ldots = \\ &= \frac{1}{2}RG_{ij} + \left(\tilde{b}^{(2)}_1 R^2 + \left(\tilde{b}^{(2)}_2 - \frac{c_1}{2}\right)\Delta R\right)G_{ij} + \tilde{b}^{(2)}_3 \nabla_i \nabla_j R + \ldots\end{aligned}$$

- Comparing two previous expressions, we find that $\tilde{b}^{(2)}_1 = \frac{1}{4}$, $\tilde{b}^{(2)}_2 = 0$ and $\tilde{b}^{(2)}_3 = 0$, therefore 2-loop $\beta$-function $\beta^{(2)}_{ij}$ is also scheme-independent.

# $\beta$-function for $D=2$ sigma models in different schemes II

- Higher order contributions to the $\beta$-function depend on the scheme, starting from the 3-loop order

$$\beta^{(3)}_{ij} = \left[ \left( \frac{5}{32} + \frac{c_1 - 2c_2}{4} \right) R^3 + \left( \frac{1}{16} - \frac{c_1 - 2c_2}{2} - (c_1^2 + c_3) \right) (\nabla R)^2 - \right.$$
$$\left. -(c_1^2 + c_3) R \nabla^2 R \right] G_{ij} - \frac{1}{16} \nabla_i R \nabla_j R - \frac{c_4}{4} \nabla_i \nabla_j \left( 3R^2 + 2\nabla^2 R \right) \ .$$

- Let us choose the covariant redefinition parameters to be

$$c_2 = -\frac{1}{16} + \frac{c_1}{2} \ , \quad c_3 = -\frac{c_1^2}{2} \, .$$

- We found the combination of the scheme change parameters, for which the $\beta$-functionup to the 4-loop order is given by

$$\beta_{ij} = \left( \frac{R}{2} + \frac{R^2}{4} + \frac{3R^3}{16} + \frac{5R^4}{32} + \frac{2+\zeta_3}{64} \nabla^2 \left( R^3 + 2R\nabla^2 R - \frac{1}{2} \nabla^2 R^2 \right) + \right.$$
$$\left. + \ldots \right) G_{ij} - \left( \frac{1}{16} + \frac{5R}{32} + \ldots \right) \nabla_i R \nabla_j R + \ldots$$

- One can notice that parts of this expression without $\zeta_3$ are the expansion of

$$\frac{R G_{ij}}{2(1-R)^{\frac{1}{2}}} - \frac{1}{16(1-R)^{\frac{5}{2}}} \nabla_i R \nabla_j R \, .$$

## "All-loop" "sausage" metric

- In (Fateev et al.'93) there was obtained the solution of 1-loop RG flow equation, which was later identified as semiclassical $\eta$-deformed $O(3)$ metric (Hoare et al.'14) (also classically integrable (Lukyanov'12)).
- All-loop metric, however, in different scheme, was conjectured in (Hoare et al.'19).
- The metric takes the form

$$ds^2 = \frac{2\kappa}{\hbar} \frac{\left(1 - \frac{\hbar\kappa\cos^2\theta}{1-\kappa^2\sin^2\theta}\right) d\theta^2 + \cos^2\theta d\chi^2}{1-\kappa^2\sin^2\theta} ,$$

where the new couplings $\hbar$ and $\kappa$ satisfy the following flow equations

$$\dot{\hbar} = 0, \quad \dot{\kappa} = \frac{\hbar(\kappa^2-1)}{2\left((1-\hbar\kappa)(1-\hbar\kappa^{-1})\right)^{\frac{1}{2}}} ,$$

and vector field has the form

$$V = \frac{\hbar}{\rho}\left\{\frac{\kappa(\kappa^2-1)\sin 2\theta}{4(1-\kappa^2\sin^2\theta)^2}, \frac{\cos^2\theta}{1-\kappa^2\sin^2\theta}\right\}, \quad \rho \overset{\text{def}}{=} \sqrt{(1-\hbar\kappa)(1-\hbar\kappa^{-1})} .$$

- We note that differential equation for $\kappa$ is uniformized by $\rho$

$$\left(\frac{1-\rho-\hbar}{1+\rho-\hbar}\right)^{1-\hbar}\left(\frac{1+\rho+\hbar}{1-\rho+\hbar}\right)^{1+\hbar} = e^{2\hbar(t-t_0)} ,$$

which resembles the integral equation from (Fateev'19).

## "All-loop" $\lambda$ model metric

- There exists a solution to the 1-loop RG flow equation without any isometries

$$ds^2 = \frac{2}{\hbar}\frac{\kappa dp^2 + \kappa^{-1} dq^2}{1 - p^2 - q^2}\,, \quad \text{where} \quad \kappa = \frac{1-\lambda}{1+\lambda}\,.$$

- This metric is one-loop renormalizable with $\kappa$ running according to the leading in $\hbar$ order of and the vector field given by

$$V_p = \frac{p}{1-p^2-q^2}\,, \quad V_q = \frac{q}{1-p^2-q^2}\,.$$

- We propose an $\hbar$ completion which is also two-loop exact similar to the all-loop "sausage" action

$$ds^2 = \frac{2}{\hbar}\left(\frac{\big(\kappa-\hbar\big)dp^2 + \big(\kappa^{-1}-\hbar\big)dq^2}{1-p^2-q^2} - \hbar\frac{\big(pdp+qdq\big)^2}{\big(1-p^2-q^2\big)^2}\right)\,.$$

supplemented by the following vector field

$$V_p = \frac{p\left(\frac{1-\hbar\kappa}{1-\hbar\kappa^{-1}}\right)^{\frac{1}{2}}}{1-p^2-q^2}\left(1 - \frac{\hbar}{2\kappa}\,\frac{1-\left(\frac{1-\kappa^2}{1-\hbar\kappa}\right)q}{1-p^2-q^2}\right)\,, \quad V_q = \{p \leftrightarrow q, \kappa \to \kappa^{-1}\}\,.$$

- Surprisingly, the parameter $\kappa$ satisfies the same RG flow differential equation as for the $\eta$-deformed model.

## UV limit of the "sausage" and $\lambda$ model metrics

- We perform the following change of the variables and the coupling constants

$$\sin\theta = \kappa^{-1}\tanh\frac{x}{2}\,, \quad \chi = \frac{y}{2} + \frac{i}{2}\log\left(1 - \frac{1-\kappa^2}{1+\kappa^2}\cosh x\right) \quad \text{for "sausage" model}\,,$$

$$p^2 + q^2 = e^{iy}\,, \qquad \frac{p^2-q^2}{p^2+q^2} = \cosh x \qquad \text{for } \lambda \text{ model}\,.$$

- Then one has

$$ds^2_{\text{sausage}} = A dx^2 + \left(A + \frac{1}{2}\right) dy^2 + B\left(e^x (dx + i dy)^2 + e^{-x}(dx - i dy)^2\right)$$

and

$$ds^2_\lambda = \frac{1}{1-e^{-iy}}\times$$
$$\times\left(A dx^2 + \frac{1 - \frac{A}{A+\frac{1}{2}}e^{-iy}}{1-e^{-iy}}\left(A + \frac{1}{2}\right) dy^2 + B\left(e^x (dx + i dy)^2 + e^{-x}(dx - i dy)^2\right)\right)\,.$$

# UV limit of the "sausage" and $\lambda$ model metrics II

- It is convenient to rescale

$$x \to \frac{x}{\sqrt{2A}}\,, \quad y \to \frac{y}{\sqrt{2A+1}}\,.$$

- Then we obtain for the sausage sigma model

$$ds^2_{\text{sausage}} = \frac{1}{2}\left(dx^2+dy^2\right) + \\ + \frac{B}{2A}\left(e^{\frac{x}{\sqrt{2A}}}\left(dx + i\frac{\sqrt{2A}}{\sqrt{2A+1}}dy\right)^2 + e^{-\frac{x}{\sqrt{2A}}}\left(dx - i\frac{\sqrt{2A}}{\sqrt{2A+1}}dy\right)^2\right)$$

and for the $\lambda$-deformed sigma model

$$ds^2_\lambda = \frac{1}{2}\left(\frac{1}{1-e^{-i\frac{y}{\sqrt{2A+1}}}}dx^2 + \frac{1-\frac{2A}{2A+1}e^{-i\frac{y}{\sqrt{2A+1}}}}{\left(1-e^{-i\frac{y}{\sqrt{2A+1}}}\right)^2}dy^2\right) + \\ + \frac{\frac{B}{2A}}{1-e^{-i\frac{y}{\sqrt{2A+1}}}}\left(e^{\frac{x}{\sqrt{2A}}}\left(dx + i\frac{\sqrt{2A}}{\sqrt{2A+1}}dy\right)^2 + e^{-\frac{x}{\sqrt{2A}}}\left(dx - i\frac{\sqrt{2A}}{\sqrt{2A+1}}dy\right)^2\right)\,.$$

## UV limit of the "sausage" and $\lambda$ model metrics III

- Coefficients $A$ and $B$ have the following $t \to -\infty$ expansion

$$A = \frac{1-\hbar}{2\hbar} + \mathcal{O}\left(e^{\frac{2\hbar(t-t_0)}{1-\hbar}}\right), \quad B = \frac{1-\hbar}{2\hbar}\hbar^{\frac{\hbar}{1-\hbar}} e^{\frac{\hbar(t-t_0)}{1-\hbar}} + \mathcal{O}\left(e^{\frac{3\hbar(t-t_0)}{1-\hbar}}\right).$$

- New parameter together with an additional imaginary shift of $y$ coordinate

$$b \overset{\text{def}}{=} \sqrt{\frac{1-\hbar}{\hbar}}, \quad y \to y + \frac{i(t-t_0)}{b^2}.$$

- We derive the following UV limits

$$ds^2_{\text{sausage}} = \frac{1}{2}\left(dx^2 + dy^2\right) +$$
$$+ e^{\frac{t-t_0}{b^2}}\left(e^{\frac{x}{b}}\left(dx + \frac{ib}{\sqrt{1+b^2}}dy\right)^2 + e^{-\frac{x}{b}}\left(dx - \frac{ib}{\sqrt{1+b^2}}dy\right)^2\right) + \mathcal{O}\left(e^{\frac{3(t-t_0)}{b^2}}\right)$$

and

$$ds^2_{\lambda} = \frac{1}{2}\left(dx^2 + dy^2\right) + e^{\frac{t-t_0}{b^2}}\left(e^{\frac{x}{b}}\left(dx + \frac{ib}{\sqrt{1+b^2}}dy\right)^2 + e^{-\frac{x}{b}}\left(dx - \frac{ib}{\sqrt{1+b^2}}dy\right)^2\right.$$
$$\left. + e^{\frac{t-t_0}{b^2}} e^{-\frac{iy}{\sqrt{1+b^2}}}\left(dx + \frac{ib}{\sqrt{1+b^2}}dy\right)\left(dx - \frac{ib}{\sqrt{1+b^2}}dy\right)\right) + \mathcal{O}\left(e^{\frac{3(t-t_0)}{b^2}}\right).$$

## “Cigar” metric

- UV limit of the $\lambda$ metric

$$ds^2 = \frac{1}{2}\left(\frac{dx^2}{1-e^{-\frac{iy}{\sqrt{1+b^2}}}} + \frac{1-\frac{b^2}{1+b^2}e^{-\frac{iy}{\sqrt{1+b^2}}}}{\left(1-e^{-\frac{iy}{\sqrt{1+b^2}}}\right)^2}dy^2\right),$$

solves the conformal equation

$$\beta_{ij}(G)+2\nabla_i\nabla_j\Phi = 0 \quad \text{with} \quad \Phi = -\frac{1}{2}\log\left(1-e^{\frac{iy}{\sqrt{1+b^2}}}\right) - \frac{1}{4(1+b^2)\left(1-e^{\frac{iy}{\sqrt{1+b^2}}}\right)}.$$

- Exchanging $x \leftrightarrow y$ and $b \leftrightarrow i\sqrt{1+b^2}$ and substituting $x = b\log(1-\coth^2 r)$ and $y = b\varphi$, we obtain the metric ($b^2 = 1/\hbar$)

$$ds^2 = \frac{2}{\hbar}\left(\left(1+\frac{\hbar}{\cosh^2 r}\right)dr^2 + \tanh^2 r d\varphi^2\right).$$

- This metric is connected to the exact cigar metric (Dijkgraaf, Verlinde, Verlinde'92)

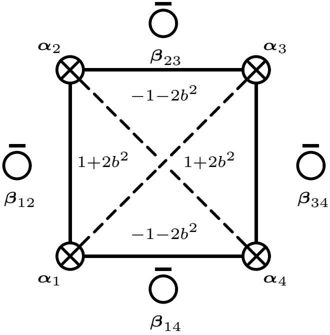
$$ds^2_{\text{DVV}} = \frac{2}{\hbar}\left(dr^2 + \frac{1}{\coth^2 r - \frac{\hbar}{\hbar+1}}d\varphi^2\right)$$

by simple covariant metric redefinition

$$(1-R)^{\frac{1}{2}}\, ds^2 = ds^2_{\text{DVV}}\left(\varphi \to \frac{\varphi}{\sqrt{1+\hbar}}\right).$$

## QFT from screening charges

- We want to check whether the metric is consistent with the screening charges corresponding to the $\eta$-deformed $O(3)$ sigma model (Fateev et al.'93).
- Let us recall that the theory in question may be determined by the following set of fermionic screenings




- By utilizing Cartesian coordinates as in (Litvinov, Spodyneiko'18) we can parametrize the fermionic screening lengths as follows

$$\boldsymbol{\alpha}_1 = bE_1 + i\beta e_1 \ , \quad \boldsymbol{\alpha}_2 = bE_1 - i\beta e_1 \ ,$$
$$\boldsymbol{\alpha}_3 = -bE_1 + i\beta e_1 \ , \quad \boldsymbol{\alpha}_4 = -bE_1 - i\beta e_1 \ .$$

# UV limit of QFT

- The model is defined by 4 fermionic screenings given by

$$\mathcal{S}_k = \oint e^{(\boldsymbol{\alpha}_k \cdot \boldsymbol{\varphi})} dz \,.$$

- In addition, there are 4 dressed screenings, whose lengths are determined by the formula

$$\boldsymbol{\beta}_{ij} = \frac{2(\boldsymbol{\alpha}_i + \boldsymbol{\alpha}_j)}{(\boldsymbol{\alpha}_i + \boldsymbol{\alpha}_j)^2}$$

  and which are given by

$$\boldsymbol{\beta}_{12} = \frac{E_1}{b} \,, \quad \boldsymbol{\beta}_{34} = -\frac{E_1}{b} \,, \quad \boldsymbol{\beta}_{13} = -i\frac{e_1}{\beta} \,, \quad \boldsymbol{\beta}_{24} = i\frac{e_1}{\beta} \,.$$

- To establish connection with the metric we can consider the gaussian action with two dressed screenings in the deep UV limit. First, let us remember that the screening charges are defined up to total derivative

$$\mathcal{S}_{ij} = \oint dz((\boldsymbol{\alpha}_i + \lambda\boldsymbol{\beta}_{ij}) \cdot \partial\boldsymbol{\varphi}) e^{(\boldsymbol{\beta}_{ij} \cdot \boldsymbol{\varphi})} \,.$$

- One can actually see that in the UV expansion of the $\eta$-deformed model there appear the dressed screenings $\boldsymbol{\beta}_{12}$ and $\boldsymbol{\beta}_{34}$, while for the $\lambda$-deformed model we have to also include $\boldsymbol{\beta}_{13}$.

# $\beta$-function of 2-dimensional supersymmetric sigma model

- The $\beta$-function is known up to 4 loops in the $\mathcal{N}=1$ case (Alvarez-Gaumé, Freedman, Mukhi'81, Alvarez-Gaumé'81, Grisaru, van de Ven, Zanon'86) and up to 5 loops in the $\mathcal{N}=2$ case (Grisaru, Kazakov, Zanon'87).

- In the minimal subtraction scheme up to 4 loops in the case of $D=2$ target space it is given by

$$\beta_{ij} = \frac{R}{2} G_{ij} + \frac{\zeta_3}{96} \nabla^2 R^3 G_{ij} + \mathcal{O}(\hbar^5)\,.$$

- There exists the scheme redefinition

$$c_2 = 0\,, \quad c_3 = -\frac{c_1^2}{2}\,, \quad c_5 = 0\,, \quad c_6 = \frac{c_1^3}{3} + \frac{c_1 c_4}{2} + \frac{5\zeta_3}{96}\,, \quad c_7 = \frac{c_1^3}{3} + \frac{5\zeta_3}{96}\,,$$
$$c_8 = \frac{c_1^3}{6} + \frac{\zeta_3}{96}\,, \quad c_9 = -3c_{11} - 6c_1 c_4\,, \quad c_{10} = -3c_{11} - 5c_1 c_4\,,$$

which allows to eliminate the 4-loop contribution to the $\beta$-function.

# $\beta$-function of 2-dimensional supersymmetric sigma model II

- In that special scheme the $\beta$-function is given by

$$\beta_{ij} = \frac{R}{2} G_{ij} + \mathcal{O}(\hbar^5)\,.$$

- Therefore, in this scheme the metric and the vector field

$$ds^2 = \frac{2\kappa}{\hbar}\frac{d\theta^2 + \cos^2\theta d\chi^2}{1-\kappa^2\sin^2\theta}\,, \quad V = \hbar\left\{\frac{\kappa(\kappa^2-1)\sin 2\theta}{4(1-\kappa^2\sin^2\theta)^2}, \frac{\cos^2\theta}{1-\kappa^2\sin^2\theta}\right\}$$

  solve the RG flow equation up to 4 loops.

- It is plausible that for arbitrary dimension of the target space there also exists the scheme with these properties.

- For example, this can be checked in the $\mathcal{N}=2$ case by switching to the description in terms of Kähler potential. We can consider the $\beta$-function for the Kähler potential $K$, which is $beta_K$ and much simpler as it is a scalar.

- It remains to interpret the scheme redefinitions in terms of the redefinitions of Kähler potential $K$.

# Conclusions and open problems

- We found the renormalization scheme, in which the expression for the 4-loop $\beta$-function for $D = 2$ sigma models is particularly simple.
- It was shown to be connected to the $\beta$-function in the minimal subtraction scheme in the first 4 loop orders by some covariant metric redefinition.
- We found the 4-loop solution to RG flow equation, corresponding to the $\eta$- and $\lambda$-deformed $O(3)$ sigma model, which was also shown to be consistent with the screening charges defining this theory.
- The renormalization scheme in question possesses an interesting property that the screenings do not receive counterterm corrections, which requires further investigation.
- Found the "cigar" metric with one exponent solves the RG flow with some certain dilaton field.
- Generalize the obtained result for higher dimensional sigma model target spaces.

Thanks for your attention!